

The University of Chicago

THE NULL FORMS $Ax^2 + By^2 + Cz^2 + Du^2$
WHICH REPRESENT ALL
INTEGERS

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1936

By
JOHN CLARK BRIXEY

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1. INTRODUCTION.

Beginning with any known solution x, y, z, u , not all zero, of the null form $F \equiv Ax^2 + By^2 + Cz^2 + Du^2 = 0$ we show 1) from this solution we can construct solutions of $F = g$, g arbitrary, or 2) from this solution we can construct solutions of $F = 0$ such that from these solutions we can obtain solutions of $F = g$, g arbitrary.

In § 3, Theorems 3 - 14, we develop necessary conditions (2)-(13) for any F to be universal. The purpose of the remainder of this paper is to prove these conditions sufficient in case F is a null form. We obtain as our

Principal Theorem. Every null form F having properties (2)-(13) is universal.

2. DEFINITIONS. THEOREMS CONCERNING SOLVABILITY OF $F = 0$.

In the sequel all letters used will be understood to represent integers. F is defined by

(1) $F = Ax^2 + By^2 + Cz^2 + Du^2$, $ABCD \neq 0$, $A = aE^2$, $B = bG^2$, $C = cH^2$, $D = dJ^2$, where E^2, G^2, H^2, J^2 are the largest squares contained in A, B, C, D respectively. F is said to be universal if $F = g$ is solvable in integers for all g . F is said to be null if $F = 0$ is solvable in integers not all zero. A proper solution x, y, z, u of $F = 0$ is one such that $(x, y, z, u) = 1$. The notation (x, y, z, u) symbolizes the greatest common positive divisor of x, y, z, u .

Theorem 1. $F = 0$ is solvable if and only if

$f \equiv ax^2 + by^2 + cz^2 + du^2 = 0$ is solvable.

Proof: To every solution x, y, z, u of $f = 0$ corresponds the solution $GHJx, EHJy, EGJz, EGHu$ of $F = 0$.

To every solution X, Y, Z, U of $F = 0$ corresponds the solution EX, GY, HZ, JU of $f = 0$.

The conditions for solvability of $f = 0$ are stated in

Theorem 2.¹ If the coefficients of $f = ax^2 + by^2 + cz^2 + du^2$ are integers, not zero, without square factors and no three of them have a common factor > 1 ,² the necessary and sufficient conditions that $f = 0$ be solvable are I, II, III.

I a, b, c, d are not all of the same sign.

II (i) $-(a, c)(a, d)(b, c)(b, d)\gamma\delta$ is a quadratic residue of every odd prime p dividing either (a, b) or (c, d) such that at the same time Legendre's symbol $(\alpha\beta\gamma\delta/p) = +1$;

(ii) $-(a, b)(a, d)(b, c)(c, d)\beta\delta$ is a quadratic residue of every odd prime p dividing either (a, c) or (b, d) for which $(\alpha\beta\gamma\delta/p) = +1$;

(iii) $-(a, b)(a, c)(b, d)(c, d)\beta\gamma$ is a quadratic residue of every odd prime p dividing either (a, d) or (b, c) for which $(\alpha\beta\gamma\delta/p) = +1$; where

1. Professor L. E. Dickson, *Studies in the Theory of Numbers*, pp. 70-76. See also R. G. Archibald, *Criteria for the Solution of a Certain Quadratic Diophantine Equation*, *Bulletin of the American Mathematical Society*, Vol. 37, (1931), pp. 608-614. L. J. Mordell, *Note on the Diophantine Equation $ax^2 + by^2 + cz^2 + dt^2 = 0$* , *Bulletin of the American Mathematical Society*, Vol. 38, (1932), pp. 277-282.
2. If a, b, c have a common factor $p > 1$ we deduce a like equation of smaller determinant $abcd/p^2$ which is solvable if and only if $f = 0$ is solvable. Repetitions lead to an equation no three of whose coefficients have a common factor > 1 .

$$a = (a,b)(a,c)(a,d)\alpha, \quad b = (a,b)(b,c)(b,d)\beta, \\ c = (a,c)(b,c)(c,d)\gamma, \quad d = (a,d)(b,d)(c,d)\delta.$$

III Either

$$(i) \quad abcd \equiv 2, 3, 5, 6, 7 \pmod{8};$$

or

$$(ii) \quad abcd \equiv 1 \text{ and } a+b+c+d \equiv 0 \pmod{8};$$

or

(iii) $abcd \equiv 4 \pmod{8}$, and, if a and b are even and c and d odd, either $abcd/4 \equiv 3, 5, 7 \pmod{8}$, or $abcd/4 \equiv 1 \pmod{8}$ and

$$\frac{a}{2} + \frac{b}{2} + c + d \equiv \frac{c^2d^2 - 1}{2} \pmod{8}.$$

3. NECESSARY CONDITIONS FOR UNIVERSALITY OF F.

We shall discover conditions restricting the coefficients of F and in subsequent sections prove that every null F subject to these restrictions is universal.

Theorem 3. If F is universal $(A, B, C, D) = 1$.

Theorem 4. If F is universal no two of A, B, C, D are divisible by 8.

Proof: (i) Let $8/A, 8/B, 2 \nmid CD$.¹ Then

$$F \equiv Cz^2 + Du^2; \quad Cz^2 \equiv 0, C, 4; \quad Du^2 \equiv 0, D, 4 \pmod{8}.$$

The even residues of F modulo 8 are 0, 4, $C+D$.

(ii) Let $8/A, 8/B, 2/C, 2/D$. Then

$$F \equiv Cz^2 + Du^2; \quad Cz^2 \equiv 0, C; \quad Du^2 \equiv 0, D, 4 \pmod{8}.$$

1. $p \nmid q$ means q is not divisible by p .
 p/q means p divides q .

The odd residues of F modulo 8 are $D, D+C$.

Theorem 5. If F is universal no three of A, B, C, D are divisible by 4.

Proof: Let $4/A, 4/B, 4/C, 2 \nmid D$. Then $F \equiv Du^2 \equiv 0, D \pmod{4}$.

Theorem 6. F is not universal when two of its coefficients are odd and the other two each divisible by 4.

Proof: Let $4/A, 4/B, 2 \nmid CD$. Then

$$F \equiv Cz^2 + Du^2; Cz^2 \equiv C, 0; Du^2 \equiv D, 0 \pmod{4}.$$

The odd residues of F modulo 4 are C, D . The even residues are $C+D, 0$.

(i) If $C \equiv D \pmod{4}$ there is only one odd residue.

(ii) If $C \not\equiv D \pmod{4}$, then $C+D \equiv 0 \pmod{4}$, and there is only one even residue.

Theorem 7. F is not universal when three of A, B, C, D are congruent modulo 4 while the fourth is $\equiv 0 \pmod{8}$.

Proof: Let $A \equiv B \equiv C \pmod{4}, 2 \nmid ABC, 1 \nmid 8/D$. Then

$$F \equiv Ax^2 + By^2 + Cz^2; Ax^2 \equiv 0, 4, A; By^2 \equiv 0, 4, B;$$

$$Cz^2 \equiv 0, 4, C \pmod{8}.$$

The odd residues of F modulo 8 are $A, B, C, A+4, B+4, C+4, A+B+C$.

(i) $A \equiv B \equiv C \pmod{8}$. The odd residues of F are $A, A+4, 3A$.

(ii) $A \equiv B \not\equiv C \pmod{8}$. Then $A+4 \equiv B+4 \equiv C$, $C+4 \equiv A \equiv B \pmod{8}$ and the odd residues of F are $A, A+4, A+B+C$.

1. Theorem 3 disposes of $2/ABC$.

Theorem 8. F is not universal when one of its coefficients is odd, two of them $\equiv 2 \pmod{4}$, and the fourth $\equiv 0 \pmod{8}$.

Proof: Let $2 \nmid A$, $B \equiv C \equiv 2 \pmod{4}$, $8 \mid D$. The odd residues of F modulo 8 are A , $A+B$, $A+C$, $A+B+C$.

(i) If $B \equiv C \pmod{8}$, then $A+B \equiv A+C \pmod{8}$.

(ii) If $B \not\equiv C \pmod{8}$, then $B+C \equiv 0$ and $A+B+C \equiv A \pmod{8}$.

Theorem 9. F is not universal when one of A , B , C , D is odd, a second $\equiv 2 \pmod{4}$, a third $\equiv 4 \pmod{8}$, and the fourth $\equiv 0 \pmod{16}$.

Proof: Let $2 \nmid A$, $B \equiv 2 \pmod{4}$, $C \equiv 4 \pmod{8}$, $16 \mid D$. The residues of F modulo 16 which are $\equiv 2 \pmod{4}$ are B , $B+4A$, $B+C$, $B+4A+C$.

(i) If $4A \equiv C \pmod{16}$, then $B+4A \equiv B+C \pmod{16}$.

(ii) If $4A \not\equiv C \pmod{16}$, then $B+4A+C \equiv B \pmod{16}$.

Theorem 10. F is not universal when two of A , B , C , D are odd and $\equiv$ modulo 8, while of the remaining, one is $\equiv 0 \pmod{16}$, and the other is $\equiv$ to the sum of the two odd coefficients modulo 8.

Proof: Let $A \equiv B$, $C \equiv 2A \pmod{8}$, $16 \mid D$, $2 \nmid AB$. The residues of F modulo 16 which are $\equiv 2 \pmod{4}$ are $A+B$, $A+B+8$, C , $C+4A \equiv C+4B$, $C+4A+4B \equiv C+8$. Either $A+B \equiv C$ and $A+B+8 \equiv C+8 \pmod{16}$ or $A+B \equiv C+8$ and $A+B+8 \equiv C \pmod{16}$.

Theorem 11. F is not universal when one of A , B , C , D is $\equiv 0 \pmod{16}$, a second $\equiv 2 \pmod{4}$, while the sum of the remaining is $\equiv 4 \pmod{8}$.

Proof: Let $A+B \equiv 4 \pmod{8}$, $2/AB, {}^1 C \equiv 2 \pmod{4}$, $16/D$.
 The residues of F modulo 16 which are $\equiv 2 \pmod{4}$ are C ,
 $C+A+B$, $C+4A$, $C+4B$, $C+A+B+8$.

Theorem 12. F is not universal when $B \equiv A+4 \pmod{8}$,
 $C \equiv 2A+4 \pmod{8}$, $16/D$.

Proof: $2/AB, {}^1$ The residues of F modulo 16 which
 are $\equiv 2 \pmod{4}$ are $A+B$, $A+B+8$, C , $C+4A \equiv C+4B$, $C+4A+4B \equiv$
 $C+8$. Either $C \equiv A+B$ and $C+4A+4B \equiv A+B+8 \pmod{16}$ or
 $C \equiv A+B+8$ and $C+4A+4B \equiv A+B \pmod{16}$.

Theorem 13. If F is universal no three of A , B , C , D
 are divisible by an odd prime.

Proof: If an odd prime $p/(A,B,C)$, then $F \equiv Du^2 \pmod{p}$
 and represents only $\frac{p+1}{2}$ incongruent residues modulo p .

Theorem 14. F is not universal if an odd prime p is
 a factor of two of its coefficients, at least one of which is
 divisible by p^2 , while the negative of the product of the two
 remaining coefficients is a quadratic non-residue of p .

Proof: Let $C = p^\gamma C'$, $D = p^\delta D'$, $p/C'D'$, $\delta \geq \gamma \geq 1$,
 $(-AB/p) = -1$. $F = Ax^2 + By^2 + p^\gamma C'z^2 + p^\delta D'u^2 \equiv Ax^2 + By^2 \pmod{p}$.
 $F \equiv 0 \pmod{p}$ implies $x \equiv y \equiv 0 \pmod{p}$. Hence

(i) If $\delta \geq \gamma > 1$, then $F \not\equiv kp \pmod{p^n}$, p/k , $n > 1$.

(ii) If $\delta > \gamma = 1$, then $F \equiv Ax^2 + By^2 + pC'z^2 \pmod{p^2}$
 and $F \equiv kp \pmod{p^2}$ is not solvable for all k since $C'z^2$ will
 take only $\frac{p+1}{2}$ incongruent values modulo p .

In view of Theorems 3-14 it follows that necessary
 conditions for F to be universal are:

1. Theorem 3 disposes of $2/AB$.

- (2) $(A, B, C, D) = 1$.
- (3) No two of A, B, C, D are divisible by 8.
- (4) No three of A, B, C, D are divisible by 4.
- (5) No two of A, B, C, D are odd while simultaneously the other two are each divisible by 4.
- (6) Three of A, B, C, D congruent modulo 4, the fourth $\equiv 0 \pmod{8}$ do not hold simultaneously.
- (7) One of A, B, C, D odd, two $\equiv 2 \pmod{4}$, and the fourth $\equiv 0 \pmod{8}$ do not hold simultaneously.
- (8) One of A, B, C, D odd, a second $\equiv 2 \pmod{4}$, a third $\equiv 4 \pmod{8}$, and the fourth $\equiv 0 \pmod{16}$ do not hold simultaneously.
- (9) No two of A, B, C, D are odd and $\equiv$ modulo 8, while of the remaining, one is $\equiv 0 \pmod{16}$, and the other is $\equiv$ to the sum of the 2 odd coefficients modulo 8.
- (10) One of $A, B, C, D \equiv 0 \pmod{16}$, a second $\equiv 2 \pmod{4}$, the sum of the remaining $\equiv 4 \pmod{8}$ do not hold simultaneously.
- (11) $B \equiv A + 4 \pmod{8}$, $C \equiv 2A + 4 \pmod{8}$, $16/D$ do not hold simultaneously.
- (12) No three of A, B, C, D are divisible by an odd prime.
- (13) No odd prime p is a factor of two of the coefficients of F , at least one of which is divisible by p^2 , while simultaneously the negative of the product of the two remaining coefficients is a quadratic non-residue of p .

In the following sections it is our purpose to prove that every null F satisfying the conditions (2)-(13) is universal.

4. SOLUTIONS OF CERTAIN DIOPHANTINE EQUATIONS.

Let x_1, y_1, z_1, u_1 be a proper solution of $F=0$ and let π be defined by

$$(14) \quad (\lambda, \mu, \nu, \rho) = \pi, \quad \lambda = Ax_1, \quad \mu = By_1, \quad \nu = Cz_1, \quad \rho = Du_1.$$

There exist ξ, η, ζ, ϕ such that

$$(15) \quad \lambda \xi + \mu \eta + \nu \zeta + \rho \phi = k\pi, \quad k \text{ any integer.}$$

We propose to find a solution of $F=g$ of the form

$$(16) \quad x = hx_1 + \xi, \quad y = hy_1 + \eta, \quad z = hz_1 + \zeta, \quad u = hu_1 + \phi.$$

Since $Ax_1^2 + By_1^2 + Cz_1^2 + Du_1^2 = 0$, $F=g$ is satisfied if

$$(17) \quad 2\pi kh = g - w,$$

where

$$(18) \quad w = A\xi^2 + B\eta^2 + C\zeta^2 + D\phi^2.$$

If $\xi_1, \eta_1, \zeta_1, \phi_1$ is a second solution of (15), write

$$X = \xi - \xi_1, \quad Y = \eta - \eta_1, \quad Z = \zeta - \zeta_1, \quad U = \phi - \phi_1.$$

We seek the general solution of

$$(19) \quad \lambda X + \mu Y + \nu Z + \rho U = 0.$$

Write (19) in the form

$$(19_1) \quad \pi\sigma(LX + MY) - \pi\tau(NZ + RU) = 0,$$

where

$$(19_2) \quad (\lambda, \mu) = \pi\sigma, \quad (\nu, \rho) = \pi\tau, \quad \lambda = \pi\sigma L, \quad \mu = \pi\sigma M, \quad \nu = -\pi\tau N, \\ \rho = -\pi\tau R, \quad (L, M) = (N, R) = (\sigma, \tau) = 1.$$

Then

$$(20) \quad LX + MY = \tau T, \quad NZ + RU = \sigma T, \quad T \text{ arbitrary.}$$

There exist integers λ', m, n, r satisfying

$$(21) \quad L\lambda' + Mm = 1, \quad Nn + Rr = 1.$$

From (20) and (21),

$$L(X - \lambda'\tau T) + M(Y - m\tau T) = 0, \quad N(Z - n\sigma T) + R(U - r\sigma T) = 0.$$

Therefore

$$(22) \quad X - \lambda \tau T = MP, \quad Y - m \tau T = -LP, \quad Z - n \sigma T = RQ, \quad U - r \sigma T = -NQ,$$

where T, P, Q are arbitrary, give general solution of (19).

Hence if $\xi_1, \eta_1, \zeta_1, \phi_1$ is one solution of (15) the general solution is

$$(23) \quad \begin{aligned} \xi &= \xi_1 + \lambda \tau T + MP, & \eta &= \eta_1 + m \tau T - LP, & \zeta &= \zeta_1 + n \sigma T + RQ, \\ \phi &= \phi_1 + r \sigma T - NQ. \end{aligned}$$

Theorem 15. Let $\xi, \eta, \zeta, \phi, k$ be variables satisfying (15), then

$$(24) \quad X = 2\pi k \xi - wx, \quad Y = 2\pi k \eta - wy, \quad Z = 2\pi k \zeta - wz, \quad U = 2\pi k \phi - wu,$$

constitute solutions of $F = 0$.

$$\begin{aligned} \text{Proof: } AX^2 + BY^2 + CZ^2 + DU^2 &= 4\pi^2 k^2 (A\xi^2 + B\eta^2 + C\zeta^2 + D\phi^2) \\ &- 4\pi kw (A\xi_1 \xi + B\eta_1 \eta + C\zeta_1 \zeta + D\phi_1 \phi) + w^2 (Ax_1^2 + By_1^2 + Cz_1^2 + Du_1^2) = 0. \end{aligned}$$

The letters and symbols used in this section retain the same definitions in the sequel. Consequently attention will be called to the place of definition only when needed for emphasis.

5. ODD PRIME FACTORS OF π .

From (17) we see that it is necessary to determine the nature of w with respect to the prime factors of π . We study first the odd prime factors of π . (12) states that no three of A, B, C, D are divisible by an odd prime if F is universal. In §5.1 we dispose of the cases in which an odd prime factor p of π is a common factor of only two of the coefficients of F , neither of which is divisible by p^2 . To do this we show that $w \equiv g \pmod{p}$, g arbitrary, is solvable. In §5.2 we show that it is possible to construct solutions of $F = 0$ from a given solution of $F = 0$ such that the corresponding π are prime

to p in case an odd prime p divides π and is a common divisor of just two of A, B, C, D at least one of which is divisible by p^2 . Consequently we are able to dispose of all cases in which an odd prime p is a divisor of π and of exactly two of the coefficients of F . In § 5.3 we consider the cases in which an odd prime divides π and but one of A, B, C, D . Again we are able to construct solutions such that the corresponding π are prime to p . In each division certain immediate extensions of results are made and theorems concerning universal null F are stated and proved.

In §§ 5.1-5.2 we shall consider the cases in which just two of the coefficients of F , say C and D , each contain an odd prime factor p and in addition x_1, y_1, z_1, u_1 is a proper solution of $F=0$ such that $\pi = p^K \pi'$, $p \nmid \pi'$, $K \geq 1$. $(C, D) = p^\psi (C', D')$, $\psi \geq 1$. $p \nmid AB$.

5.1. $\psi = 1$, $p \nmid C'D'$.

Theorem 16. There exist ξ, η, ζ, ϕ satisfying (15) and such that $w = A\xi^2 + B\eta^2 + C\zeta^2 + D\phi^2 \equiv g \pmod{p}$ is solvable for arbitrary g , where p is an odd prime factor of π and a common factor of exactly two of the coefficients of F neither of which is divisible by p^2 .

Proof: Let $C = pC'$, $D = pD'$, $p \nmid ABC'D'$. Applying (23),

$$w \equiv A\xi^2 + B\eta^2 \equiv A(\xi_1 + \tau T + MP)^2 + B(\eta_1 + m\tau T - LP)^2 \pmod{p}.$$

But

$$\xi_1 + \tau T + MP \equiv g_1, \quad \eta_1 + m\tau T - LP \equiv g_2 \pmod{p}$$

are solvable for arbitrary g_1 and g_2 . $(\tau, p) = (L, M) = 1$. Hence the theorem results on applying

Theorem 17.¹ If p is an odd prime dividing neither A nor B and if g is any integer, $Ax^2 + By^2 \equiv g \pmod{p}$ has integral solutions.

Theorem 18. $w \equiv g \pmod{\pi'}$, $\pi = 2^e \pi'$, $2 \nmid \pi'$, is solvable for arbitrary g if π' is the product of distinct odd primes p_i , p_i^2 not dividing any one of A, B, C, D and (12) holding.

Proof: Theorem 16 declares $w \equiv g \pmod{p}$ is solvable for every odd prime factor p of π' under these hypotheses, g arbitrary. π' is a product of distinct odd primes. Hence the theorem.

Theorem 19. Every null form F satisfying (12) is universal if A, B, C, D are odd and contain no square factors > 1 .

Proof: 1) In (15) take $k=1$. Then $w \equiv 1 \pmod{2}$.

2.1) If $2 \nmid x, y, z, u$, $2 \nmid \lambda \mu \nu \rho$, take $k=2$. From (23) we obtain

$\xi \equiv \xi_1 + \lambda T + P$, $\eta \equiv \eta_1 + mT + P$, $\zeta \equiv \zeta_1 + nT + Q$, $\phi \equiv \phi_1 + rT + Q \pmod{2}$. Since $(\lambda, m) = (n, r) = 1$, we may choose T, P, Q modulo 2 such that any two of ξ, η, ζ, ϕ are odd, the remaining even, or such that all are even (or all odd) and obtain at will $w \equiv 2$ or $w \equiv 0 \pmod{4}$.

2.2) If $2 \nmid x, y$, $2 \nmid z$, $2 \nmid u$, $2 \nmid \lambda \mu$, $2 \nmid \tau$. We may suppose, to be definite, $2 \nmid Nn$. Take $k=1$. From (15) and (23) we obtain

$$\xi + \eta \equiv 1, \zeta \equiv \zeta_1 + T + RQ, \phi \equiv \phi_1 + rT + Q \pmod{2}.$$

1. L. E. Dickson, Annals of Mathematics, Series 2, Vol. 28, (1927), p. 333.

We may choose T, Q modulo 2 such that $\zeta \equiv \phi$ or $\zeta \not\equiv \phi \pmod{2}$. Then $w \equiv 1$ or $w \equiv 0 \pmod{2}$ respectively.

On applying Theorem 18 we find $w \equiv g \pmod{2\pi k}$ solvable for arbitrary g by proper choice of $k=1$ or 2. Now (17) is solvable.

Theorem 20. Every null F with $2 \nmid ABCD$ and satisfying (12) and (13) is universal if there exist solutions x_i, y_i, z_i, u_i of $F=0$ such that π is either equal to 1 or is a product of distinct odd primes p_i, p_i^2 dividing no one of A, B, C, D .

The proof is identical with that of Theorem 19 and consequently not repeated.

Theorem 21. Every null form F satisfying (2) and (12) is universal if A, B, C, D contain no square factors >1 .

Proof: By Theorem 18, $w \equiv g \pmod{\pi'}$ is solvable for arbitrary g where $\pi = 2^e \pi', 2 \nmid \pi'$. It remains to determine T, P, Q, k modulo 2 or 4 so that $w \equiv g \pmod{2\pi k}$ will be solvable.

I. $2 \nmid ABCD$ disposed of in Theorem 19.

II. $2 \nmid ABC, 2 \nmid D, 2 \nmid \pi$. $x_i + y_i + z_i \equiv 0 \pmod{2}$. We may consider $2 \nmid x_i, z_i, 2 \nmid y_i$, since this offers no restriction. Then $2 \nmid \lambda \vee \sigma \tau \nmid n, 2 \nmid M, 2 \nmid R$. From (15), (18) we get $\xi + \zeta \equiv k, w \equiv \xi + \eta + \zeta \pmod{2}$. From (23) we obtain
 (25) $\xi \equiv \xi_i + T, \eta \equiv \eta_i + mT + P, \zeta \equiv \zeta_i + T, \phi \equiv \phi_i + rT + Q \pmod{2}$.
 Take $k=1$, then $\xi + \zeta \equiv 1$ and $w \equiv \eta + 1 \pmod{2}$. We obtain $w \equiv 1$ or $0 \pmod{2}$ according to $P + mT + \eta_i \equiv 0$ or $1 \pmod{2}$.

III. $2 \nmid AC, B \equiv D \equiv 2 \pmod{4}$.

1) $2 \nmid \pi, 2 \nmid x_i, z_i$. Then $2 \nmid \lambda \vee \sigma \tau \nmid n, 2 \nmid M, 2 \nmid R$.

From (23) we obtain (25). From (15) and (18) we get $w \equiv \xi + \zeta \equiv k \pmod{2}$.

a) Take $k=1$. Then $w \equiv 1 \pmod{2}$.

b) Take $k=2$. Then $\xi + \zeta \equiv 0 \pmod{2}$. We may choose T, P, Q such that $T + \xi_1 \equiv T + \zeta_1 \equiv 1$ or 0 , $P + mT + \eta_1 \equiv 1$ or 0 , $Q + rT + \phi_1 \equiv 1$ or $0 \pmod{2}$ as needed to obtain $w \equiv 2$ or $0 \pmod{4}$.

2) $2/\pi, 2/y, u_1, 2/x_1, 2/z_1$. Then $2/\sigma \tau MR$. Since $2/MR$, we may select m and r both odd to satisfy (21). From (15) and (18) we get $\frac{1}{2} x_1 \xi + \eta + \frac{1}{2} z_1 \zeta + \phi \equiv k, w \equiv \xi + \zeta \pmod{2}$. (23) yields

$$\xi \equiv \xi_1 + \lambda T + P, \eta \equiv \eta_1 + T + LP, \zeta \equiv \zeta_1 + nT + Q, \phi \equiv \phi_1 + T + nQ \pmod{2}.$$

$$2.1) x_1 \equiv z_1 \equiv 2 \pmod{4}, 2/\lambda n, 2/\lambda, 2/n.$$

$$2.11) A + C \equiv 2 \pmod{4}.$$

a) Take $k=1$. Then $\xi + \eta + \zeta + \phi \equiv 1 \pmod{2}$.

1) We may choose T, P, Q modulo 2 such that $2/\xi \eta \zeta$. Then $2/\phi$ and $4/w$.

2) We may take T, P, Q modulo 2 such that $2/\xi, 2/\eta, 2/\zeta$. Then $2/\phi$ and $w \equiv 2 \pmod{4}$.

3) We may take T, P, Q modulo 2 such that $2/\xi, 2/\eta, 2/\zeta$. Then $2/\phi$ and $w \equiv A \pmod{4}$.

b) Take $k=2$. Then $\xi + \eta + \zeta + \phi \equiv 0 \pmod{2}$.

1) We may take T, P, Q modulo 4 such that $2/\xi \eta, \zeta \equiv 0$ or $2 \pmod{4}$. Then $2/\phi$ and $w \equiv A + B$ or $w \equiv A + B + 4 \pmod{8}$ according as $\zeta \equiv 0$ or $\zeta \equiv 2 \pmod{4}$.

$$2.12) A + C \equiv 0 \pmod{4}.$$

a) Take $k=1$. Then $\xi + \eta + \zeta + \phi \equiv 1 \pmod{2}$.

1) We may take T, P, Q modulo 2 such that $2 \nmid \xi \eta \zeta$. Then $2/\phi$ and $w \equiv 2 \pmod{4}$.

2) We may take T, P, Q modulo 2 such that $2 \nmid \xi, 2/\eta, 2/\zeta$. Then $2/\phi$ and $w \equiv A \pmod{4}$.

3) We may take T, P, Q modulo 2 such that $2/\xi, 2/\eta, 2/\phi$. Then $2 \nmid \zeta$ and $w \equiv C \pmod{4}$.

b) Take $k=2$. Then $\xi + \eta + \zeta + \phi \equiv 0 \pmod{2}$. We may take T, P, Q modulo 4 such that $\xi \equiv 2$ or $0 \pmod{4}$, $2 \nmid \eta, 4/\zeta$. Then $2 \nmid \phi$ and $w \equiv 4$ or $0 \pmod{8}$ according to choice of ξ .

2.2) $x_1 \equiv 0, z_1 \equiv 2 \pmod{4}, B+D \equiv 4 \pmod{8}, 2/L, 2 \nmid Nm, 2/n$.

a) Take $k=1$. Then $\eta + \zeta + \phi \equiv 1 \pmod{2}$.

1) We may take T, P, Q modulo 2 such that $2 \nmid \xi \eta, 2/\zeta$. Then $2/\phi$ and $w \equiv A+2 \pmod{4}$.

2) We may take T, P, Q modulo 2 such that $2/\xi, 2 \nmid \eta, 2/\zeta$. Then $2/\phi$ and $w \equiv 2 \pmod{4}$.

b) Take $k=2$. Then $\eta + \zeta + \phi \equiv 0 \pmod{2}$.

1) We may take T, P, Q modulo 4 such that $2 \nmid \xi, 2/\eta, \zeta \equiv 0$ or $2 \pmod{4}$. Then $2/\phi$ and $w \equiv A$ or $A+4 \pmod{8}$ according as $\zeta \equiv 0$ or $2 \pmod{4}$.

2) We may take T, P, Q modulo 4 such that $\xi \equiv 0$ or $2 \pmod{4}, 2 \nmid \eta, 4/\zeta$. Then $2 \nmid \phi$ and $w \equiv 4$ or $0 \pmod{8}$ according as $\xi \equiv 0$ or $2 \pmod{4}$.

2.3) $4/x_1, 4/z_1, B+D \equiv 0 \pmod{16}, 2/L, 2/N$.

a) Take $k=1$. Then $\eta + \phi \equiv 1 \pmod{2}$.

1) We may take T, P, Q modulo 2 such that

$2 \nmid \xi \eta$, $2 \mid \xi$. Then $2 \nmid \phi$ and $w \equiv A+2 \pmod{4}$.

2) We may take T, P, Q modulo 2 such that $2 \mid \xi$, $2 \nmid \eta$, $2 \mid \xi$. Then $2 \nmid \phi$ and $w \equiv 2 \pmod{4}$.

b) Take $k=2$. Then $\eta + \phi \equiv 0 \pmod{2}$.

1) We may take T, P, Q modulo 4 such that $2 \nmid \xi$, $2 \mid \eta$, $\xi \equiv 0$ or $2 \pmod{4}$. Then $2 \nmid \phi$ and $w \equiv A$ or $A+4 \pmod{8}$ according as $\xi \equiv 0$ or $2 \pmod{4}$.

2) We may take T, P, Q modulo 4 such that $\xi \equiv 2$ or 0 , $\xi \equiv 0 \pmod{4}$, $2 \mid \eta$. Then $2 \nmid \phi$ and $w \equiv 4$ or $0 \pmod{8}$ according as $\xi \equiv 2$ or $0 \pmod{4}$.

IV. $2 \nmid A$, $B \equiv C \equiv D \equiv 2 \pmod{4}$. Hence $2 \mid x_i, y_i + z_i + u_i \equiv 0 \pmod{2}$, $\Pi = 2\Pi'$, $2 \nmid \Pi'$. There is no loss of generality in considering $2 \mid y_i$, $2 \nmid z_i, u_i$. Then $2 \nmid \tau \text{NR}$. From (15) we obtain $\frac{1}{2} x, \xi + \zeta + \phi \equiv k \pmod{2}$. (18) and (23) yield

$$w \equiv \xi, \xi \equiv \xi_i + \lambda T + MP, \eta \equiv \eta_i + mT + LP, \zeta \equiv \zeta_i + n\sigma T + Q, \\ \phi \equiv \phi_i + r\sigma T + Q \pmod{2}.$$

1) $4 \mid x_i$, $2 \mid \sigma$. Take $k=1$. Then $\zeta + \phi \equiv 1 \pmod{2}$. We may take T, P, Q modulo 4 such that $\xi \equiv 2 \pmod{4}$, $2 \nmid \eta \zeta$ since $L\lambda + Mm = 1$. Then $2 \nmid \phi$ and $4 \mid w$. It follows that $X_i = \frac{1}{4}X$, $Y_i = \frac{1}{4}Y$, $Z_i = \frac{1}{4}Z$, $U_i = \frac{1}{4}U$, where X, Y, Z, U are defined by (24), are solutions of $F=0$ having $X_i \equiv 2 \pmod{4}$, $2 \nmid Y_i$, and only one of Z_i, U_i odd. Consequently we need to consider only

2) $x_i \equiv 2 \pmod{4}$, $2 \mid y_i$, $2 \nmid z_i, u_i$. Then $2 \mid M$, $2 \nmid \sigma L\lambda$. Take $k=1$. Then $\xi + \zeta + \phi \equiv 1 \pmod{2}$.

2.1) We may take T, P modulo 2 such that $2 \nmid \xi \eta$. Then $\zeta \equiv \phi \pmod{2}$ and $w \equiv A+2 \pmod{4}$.

2.2) We may take T, P modulo 2 such that $2 \nmid \xi$, $2 \mid \eta$.

Then $\xi \equiv \phi \pmod{2}$ and $w \equiv A \pmod{4}$.

2.3) We may take T, P, Q modulo 2 such that $2/\xi, 2/\eta, 2/\zeta$. Then $2 \nmid \phi$ and $w \equiv 2 \pmod{4}$.

2.4) We may take T, P, Q modulo 2 such that $2/\xi, 2 \nmid \eta, 2/\zeta$. Then $2 \nmid \phi$ and $w \equiv 0 \pmod{4}$.

Thus we have demonstrated that in every case $w \equiv g \pmod{2\pi k}$, $k=1$ or 2 as needed, is solvable for arbitrary g .

Theorem 22. Every null F satisfying (2), (12), (13) and having no one of A, B, C, D divisible by 4 is universal if there exist solutions x, y, z, u of $F=0$ such that $\pi = 2^e \pi'$, where π' is either equal to 1 or is a product of distinct odd primes p_i, p_i^2 dividing no one of A, B, C, D .

The proof is identical with that of Theorem 21 and consequently not repeated.

5.2. $\psi \geq 1$, AT LEAST ONE OF C, D DIVISIBLE BY p^2 .

Later in this section it is necessary to use

Theorem 23. $Ax^2 + By^2 \equiv tp \pmod{p^n}$ is solvable for arbitrary n and has solutions $x=\xi, y=\eta$, where ξ and η are prime to an odd prime p when $(-AB/p)=1$.

Proof: There are solutions $x=\xi, y=\eta, p \nmid \xi\eta$, when $n=1$ since $(-AB/p)=1$. To proceed by induction on n , let $Ax^2 + By^2 \equiv tp \pmod{p^m}$ have solutions $\xi, \eta, p \nmid \xi\eta$. Write $A\xi^2 + B\eta^2 = tp + qp^m, x = \xi + p^m X, y = \eta + p^m Y$. Then

$$\begin{aligned} Ax^2 + By^2 &= tp + qp^m + 2p^m(A\xi X + B\eta Y) + p^{2m}(AX^2 + BY^2) \\ &\equiv tp + p^m(2A\xi X + 2B\eta Y + q) \pmod{p^{m+1}}. \end{aligned}$$

Since $p \nmid \xi\eta$, we can find X, Y such that $2A\xi X + 2B\eta Y + q \equiv 0 \pmod{p}$. Then $Ax^2 + By^2 \equiv tp \pmod{p^{m+1}}$ and the induction is complete.

Theorem 24. If F is null and (13) holds, there exist solutions of $F=0$ such that $p \nmid \pi$, where p is an odd prime factor of exactly two of the coefficients of F at least one of which is divisible by p^2 .

Proof: Assume $C = p^t C'$, $D = p^v D'$, $p \nmid ABC'D'$, $v \geq t \geq 1$, $v > 1$. Let $x_1 = p^k x'$, $y_1 = p^k y'$, z_1 , u_1 be a proper solution of $F=0$ such that $\pi = p^K \pi'$, $p \nmid \pi'$, $K \geq 1$. We may assume $p \nmid u_1$, $x_1, y_1 \neq 0$.

I. $K < t$ implies $p \nmid \tau$. We may assume $p \nmid x'$. Then $p \nmid \sigma L$ and $\sigma L \xi + \sigma M \eta \equiv k \pmod{p}$. Examination of (23) shows that we may take P modulo p such that η will take any residue modulo p , which is prime to p . Now k may be chosen modulo p to take any residue which is prime to p . Hence, corresponding to any choice of η modulo p , $p \nmid \eta$, it is possible to obtain $\xi \equiv$ to any residue prime to p modulo p . By hypothesis (13) holds. Therefore it is always possible to select P , k modulo p , $p \nmid k$, such that $p \nmid \xi \eta$ and $A\xi^2 + B\eta^2 \equiv 0 \pmod{p}$. Now Theorem 23 indicates that T , P , k modulo p^e may be selected in accord with earlier choice of P , k modulo p , $p \nmid k$, such that $p \nmid \xi \eta$ and $A\xi^2 + B\eta^2 \equiv 0 \pmod{p^e}$, $e \geq K$. Then $w \equiv 0 \pmod{p^K}$.

II. $K = t$. Then $p^K(Ax'^2 + By'^2) + C'z_1^2 + p^{v-K}D'u_1^2 = 0$.

1) $v - K > 0$ implies $p \nmid z_1$, and not both x' , y' divisible by p . Therefore, as in I, we may take T , P , k , $p \nmid k$, such that $p \nmid \xi \eta$ and $w \equiv 0 \pmod{p^K}$.

2) $v - K = 0$ implies $p \nmid z_1, \tau$. Take $k = k'$, $p \nmid k'$. Since $p \nmid \tau$ and $Lx' + My' = 1$, we may take T , P modulo p such that $p \nmid \xi \eta$

and $A\xi^2 + B\eta^2 \equiv 0 \pmod{p}$. Theorem 23 indicates that T, P modulo p^e may be selected in accord with earlier choice of T, P modulo p such that $p \nmid \xi\eta$ and $A\xi^2 + B\eta^2 \equiv 0 \pmod{p^e}$, $e \geq K$. Then $w \equiv 0 \pmod{p^K}$.

III. $K > t$. Write $z_1 = p^{K-t} z'$.

1) $v = 2K - t$ implies $p \nmid z' \tau Nn$, p^{K-t}/R .

1.1) p prime to at least one of x', y' implies $p \nmid \sigma$.

Examination of (23) shows that we may take T such that p^{K-t}/ζ . Since k may be chosen any integer, it follows that we may select P, k modulo p , $p \nmid k$, such that $p \nmid \xi\eta$ and $A\xi^2 + B\eta^2 \equiv 0 \pmod{p}$. Theorem 23 indicates that T, P, k may be selected, in addition, such that $A\xi^2 + B\eta^2 \equiv 0 \pmod{p^e}$, $p \nmid \xi\eta$, $e \geq 2K - t$. Then $w \equiv 0 \pmod{p^{2K-t}}$.

1.2) $x' = p^\alpha x''$, $y' = p^\beta y''$, $p \nmid x'' y''$, $\beta \geq \alpha \geq 1$. Then $\sigma = p^\alpha \sigma'$, $p \nmid \sigma'$, $p \nmid L$.

1.21) $\alpha < K - t$. Take $k = p^\alpha k'$, $p \nmid k'$. Then p^α/ζ . (23) shows that we may take T such that p^{K-t}/ζ . Now $\sigma' L \xi + \sigma' M \eta \equiv k' \pmod{p}$. Since k' may be chosen any integer, it follows that we may select P, k' modulo p , $p \nmid k'$, such that $p \nmid \xi\eta$ and $A\xi^2 + B\eta^2 \equiv 0 \pmod{p}$. Theorem 23 indicates that T, P, k may be selected, in addition, such that $A\xi^2 + B\eta^2 \equiv 0 \pmod{p^e}$, $p \nmid \xi\eta$, $e \geq 2K - t$. Then $w \equiv 0 \pmod{p^{2K-t}}$.

1.22) $\alpha \geq K - t$. Take $k = p^{K-t} k'$, $p \nmid k'$. Then p^{K-t}/ζ . Since $p \nmid \tau$ and $LX + Mm = 1$, we may take T, P modulo p such that $p \nmid \xi\eta$ and $A\xi^2 + B\eta^2 \equiv 0 \pmod{p}$. Theorem 23 indicates that T, P may be selected, in addition, such that $A\xi^2 + B\eta^2 \equiv 0 \pmod{p^e}$, $p \nmid \xi\eta$, $e \geq 2K - t$. Then $w \equiv 0 \pmod{p^{2K-t}}$.

2) $v > 2K-t$ implies p/z' . We may assume $p \nmid x'$. Then $p \nmid \sigma L$, p/τ . Since $p \nmid \sigma$ and $(n, R)=1$, we may take T, Q such that p^{K-t}/ξ . Since k may be chosen any integer, it follows that we may select P, k modulo p , $p \nmid k$, such that $p \nmid \xi \eta$ and $A\xi^2 + B\eta^2 \equiv 0 \pmod{p}$. Theorem 23 indicates that T, P, k may be selected, in addition, such that $A\xi^2 + B\eta^2 \equiv 0 \pmod{p^e}$, $p \nmid \xi \eta$, $e \geq 2K-t$. Then $w \equiv 0 \pmod{p^{2K-t}}$.

Hence, in any case on applying (24), we obtain solutions of $F=0$ of type

$$(26) \quad X = 2p^{K+\theta} \pi' k' \xi - p^{K+\theta} \pi' k' \eta - p^{K+\theta} \pi' k' \zeta, \\ Y = 2p^{K+\theta} \pi' k' \eta - p^{K+\theta} \pi' k' \zeta - p^{K+\theta} \pi' k' \phi, \\ Z = 2p^{K+\theta} \pi' k' \zeta - p^{K+\theta} \pi' k' \phi - p^{K+\theta} \pi' k' \psi, \\ U = 2p^{K+\theta} \pi' k' \phi - p^{K+\theta} \pi' k' \psi - p^{K+\theta} \pi' k' \chi,$$

where $k = p^\theta k'$, $p \nmid k'$, $\theta = 0, \alpha, K-t$ as demanded. Since $K+\theta \leq 2K-t$, $p \nmid \xi \eta$, p^{2K-t}/w , we obtain

$$(27) \quad X_1 = \frac{1}{p^{K+\theta}} X, \quad Y_1 = \frac{1}{p^{K+\theta}} Y, \quad Z_1 = \frac{1}{p^{K+\theta}} Z, \quad U_1 = \frac{1}{p^{K+\theta}} U,$$

solutions of $F=0$ with $p \nmid X_1, Y_1$.

Theorem 25. If F is null and (12) and (13) hold, there exist solutions of $F=0$ such that π contains no odd prime factors p which are divisors of any two of the coefficients of F at least one of which is divisible by p^2 .

Proof: If there is only one such odd prime p , Theorem 24 disposes of it. If there are two or more such distinct odd primes $p_1, p_2, \dots$, then according to Theorem 24 solutions X_1, Y_1, Z_1, U_1 of $F=0$ may be constructed such that $p_1 \nmid \pi$. The theorem follows if $p_2, \dots$ are prime to π at the same time. Otherwise, from the solutions X_1, Y_1, Z_1, U_1 , solutions X_2, Y_2, Z_2, U_2 of $F=0$ may be constructed such that $p_2 \nmid \pi$. In

addition the solutions X_2, Y_2, Z_2, U_2 may be constructed such that $p_1 \nmid \pi$. This is always possible since $p_1 \nmid p_2$. In order to show this more fully we may assume $p_1 \nmid (C, D)$. Then $p_1 \nmid \tau$, $p_1 \nmid \sigma LM$, $\sigma L\xi + \sigma M\eta \equiv k \pmod{p_1}$. Take k prime to p_1 . We may take P modulo p_1 such that $p_1 \nmid \xi$. Then $p_1 \nmid \eta w$. Repetition of this procedure leads to the theorem.

Theorem 26. Every null F having $2 \nmid ABCD$ and satisfying (12) and (13) is universal if at the same time p^2 divides one of A, B, C, D another is divisible by p , p an odd prime.

Proof: This theorem is a consequence of Theorems 25, 20, 18.

Theorem 27. Every null F satisfying (2), (12), (13) is universal if no one of A, B, C, D is divisible by 4 and if at the same time p^2 divides one of A, B, C, D another is divisible by p , p an odd prime.

Proof: This theorem is a consequence of Theorems 25, 22, 18.

5.3. ODD PRIMES WHICH DIVIDE π AND JUST ONE OF A, B, C, D .

In this section we shall study the case of an odd prime factor p which divides π and just one of the coefficients of F .

We shall apply

Theorem 28.¹ If no one of a, b, c is divisible by the odd prime p , $f = ax^2 + by^2 + cz^2 \equiv k \pmod{p}$ has solutions with

1. L. E. Dickson, Annals of Mathematics, Series 2, Vol. 28, (1927), pp. 333-334.

x and y not both divisible by p .

Theorem 29.¹ If p is an odd prime not dividing abc , $f = ax^2 + by^2 + cz^2 \equiv k \pmod{p^n}$ has solutions when k and n are arbitrary.

We prove

Theorem 30. If F is null, there exist solutions such that $p \nmid \pi$, where p is an odd prime factor of just one of the coefficients of F .

Proof: Assume $D = p^e D'$, $p \nmid ABCD'$, $e \geq 2$.² Let $x_i = p^K x'$, $y_i = p^K y'$, $z_i = p^K z'$, u_i be a proper solution of $F = 0$ such that $\pi = p^K \pi'$, $p \nmid \pi'$, $K \geq 1$. (15) becomes

$$(28) \quad Ax'\xi + By'\eta + Cz'\zeta + p^{e-K} D'u_i \phi = \pi'k,$$

where at least one of x' , y' , z' is prime to p and $e \geq 2K$.

I. $e > 2K$ implies no two of x' , y' , z' are divisible by p . We may assume $p \nmid x'z'$. Then $p \nmid \sigma \tau L N n$, $p^{e-K/R}$. $Ax'^2 + By'^2 + Cz'^2 + p^{e-2K} D'u_i^2 = 0$ implies $Ax'^2 + By'^2 + Cz'^2 \equiv 0 \pmod{p}$.

(23) yields

$$\xi \equiv \xi_1 + \lambda \tau T + MP, \quad \eta \equiv \eta_1 + m \tau T - LP, \quad \zeta \equiv \zeta_1 + n \sigma T,$$

$$\phi \equiv \phi_1 + r \sigma T - NQ \pmod{p}.$$

Take $k = pk'$, $p \nmid k'$. We may take T, P modulo p such that $\xi \equiv x'$, $\eta \equiv y' \pmod{p}$ since $p \nmid \tau$ and $L\lambda + Mm = 1$. Then $\zeta \equiv z' \pmod{p}$ and $w \equiv 0 \pmod{p}$. Theorem 29 declares T, P, Q may be chosen modulo p^α such that $\xi \equiv x'$, $\eta \equiv y'$, $\zeta \equiv z' \pmod{p}$ and $w \equiv 0 \pmod{p^\alpha}$, $\alpha \geq e$.

1. L. E. Dickson, Annals of Mathematics, Series 2, Vol. 28, (1927), pp. 333-334.

2. $p \nmid \pi$ if $e < 2$.

II. $e=2K$

1) $p \nmid x'y'z'u$, implies $p \nmid \sigma \tau LMNn$, p^K/R . $C\zeta^2$ can be made to take any of $\frac{p+1}{2}$ incongruent values modulo p by choice of T . $A\xi^2 + B\eta^2$ will take $\frac{p+1}{2}$ incongruent values modulo p by choice of P corresponding to each selection for T . Hence $w \equiv A\xi^2 + B\eta^2 + C\zeta^2 \equiv 0 \pmod{p}$ for at least one choice of T, P . If $k=k'$, $p \nmid k'$, we may assume $p \nmid \xi$ as ξ, η, ζ are not all divisible by p .

2) $p \nmid x'z'u$, p/y' imply $p \nmid \sigma \tau LN\bar{x}n$, p/M , p^K/R .

(23) yields

$$\xi \equiv \xi_1 + \lambda \tau T, \quad \eta \equiv \eta_1 + m \tau T - LP, \quad \zeta \equiv \zeta_1 + n \sigma T,$$

$$\phi \equiv \phi_1 + r \sigma T - NQ \pmod{p}.$$

Therefore $A\xi^2 + C\zeta^2$ can be made to take any of $\frac{p+1}{2}$ incongruent values modulo p by choice of T . $B\eta^2$ will take $\frac{p+1}{2}$ incongruent values modulo p by choice of P corresponding to each selection of T . Hence $w \equiv A\xi^2 + B\eta^2 + C\zeta^2 \equiv 0 \pmod{p}$ for at least one choice of T, P . If $k=k'$, $p \nmid k'$, we may assume $p \nmid \xi$ as ξ, ζ are not both divisible by p .

3) $p \nmid x'u$, p/y' , p/z' imply $p \nmid \sigma L\bar{x}$, p/τ , p/M .

(23) yields

$$\xi \equiv \xi_1, \quad \eta \equiv \eta_1 - LP, \quad \zeta \equiv \zeta_1 + n \sigma T + RQ, \quad \phi \equiv \phi_1 + r \sigma T - NQ \pmod{p}.$$

Therefore $A\xi^2 + B\eta^2$ can be made to take any of $\frac{p+1}{2}$ incongruent values modulo p by choice of P . According as n or R not divisible by p , we may select T or Q such that $C\zeta^2$ will take $\frac{p+1}{2}$ incongruent values modulo p corresponding to each selection of P . Hence $w \equiv A\xi^2 + B\eta^2 + C\zeta^2 \equiv 0 \pmod{p}$ for at least one choice of T, P, Q . If $k=k'$, $p \nmid k'$ then $p \nmid \xi$.

Theorem 29 declares T, P, Q may be chosen modulo p^α such that $w \equiv 0 \pmod{p^\alpha}$, $\alpha \geq e$ in cases II.

Therefore, in any case on applying (24), we obtain solutions of $F=0$ of type

$$(29) \quad X = 2p^{k+\theta} \pi' k' \xi - p^k w x', \quad Y = 2p^{k+\theta} \pi' k' \eta - p^k w y', \\ Z = 2p^{k+\theta} \pi' k' \zeta - p^k w z', \quad U = 2p^{k+\theta} \pi' k' \phi - w u_1,$$

where $k = p^\theta k'$, $p \nmid k'$, $\theta = 0$ or 1 according as $e = 2K$ or $e > 2K$.

Since $p \nmid \xi$, $w \equiv 0 \pmod{p^\alpha}$, $\alpha \geq e$, we obtain

$$(30) \quad X_1 = \frac{1}{p^{k+\theta}} X, \quad Y_1 = \frac{1}{p^{k+\theta}} Y, \quad Z_1 = \frac{1}{p^{k+\theta}} Z, \quad U_1 = \frac{1}{p^{k+\theta}} U,$$

solutions of $F=0$ with $p \nmid X_1$.

Theorem 31. If F is null and (12) and (13) hold, there exist solutions such that π contains no odd prime factors p when p^2 is a divisor of at least one of the coefficients of F .

Proof: Theorem 25 disposes of all cases in which any two of A, B, C, D have a common odd prime factor while simultaneously at least one of the two coefficients is divisible by the square of the prime.

If there is only one such odd prime p , Theorems 25 or 30 dispose of it. If there are two or more such distinct odd primes $p_1, p_2, \dots$, then according to Theorems 24 and 30 solutions X_1, Y_1, Z_1, U_1 of $F=0$ may be constructed such that $p_1 \nmid \pi$. The theorem follows if $p_2, \dots$ are prime to π at the same time. Otherwise, from the solutions X_1, Y_1, Z_1, U_1 , solutions X_2, Y_2, Z_2, U_2 of $F=0$ may be constructed such that $p_2 \nmid \pi$. In addition the solutions X_2, Y_2, Z_2, U_2 may be constructed such that $p_1 \nmid \pi$. This is always possible since

$p_1 \nmid p_2$. In order to show this more fully we may assume $p_1 \nmid D$, $p_1 \nmid ABC$, or $p_1 \nmid (C, D)$, $p_1 \nmid AB$, corresponding to the cases in which p_1 is a divisor of one or two of the coefficients of F . There is no loss in assuming $p_1 \nmid X, Y$. Then $p_1 \nmid \sigma LM$.

1) $p_1 \nmid \tau$. Then $\sigma L\xi + \sigma M\eta \equiv k \pmod{p_1}$. Take k prime to p_1 . Now we may take T, P, Q modulo p_1 such that $p_1 \nmid \xi$, $p_1 \nmid \eta$. Then $p_1 \nmid \eta w$.

2) $p_1 \nmid \tau$. Then for any k it is possible to take T, P, Q modulo p_1 such that $p_1 \nmid \xi$, $p_1 \nmid \eta$, $p_1 \nmid \zeta$. Then $p_1 \nmid w$.

Repetition of this procedure leads to the theorem.

Theorem 32. Every null F having $2 \nmid ABCD$ and satisfying (12) and (13) is universal.

Proof: This theorem is a consequence of Theorems 31, 20, 18.

Theorem 33. Every null F having no one of A, B, C, D divisible by 4 and satisfying (2), (12) and (13) is universal.

Proof: This theorem is a consequence of Theorems 31, 22, 18.

6. STUDY OF w MODULO 2^6 , $2 \nmid \pi k \equiv 2^6 \pi'k'$, $2 \nmid \pi'k'$.

The purpose of this section is to determine the nature of w modulo 2^6 such that $w \equiv g \pmod{2^6}$ is solvable for arbitrary g . Hence we discover conditions for (17) to be solvable for h corresponding to any g . We dispose, in the order mentioned, in §§ 6.1, 6.2 and 6.3 of the cases in which exactly one, exactly two and exactly three of A, B, C, D are even. In each division certain immediate extensions of results are made and theorems concerning universal null F are stated and

proved.

6.1. JUST ONE OF A, B, C, D EVEN.

We prove a preliminary theorem.

Theorem 34. $F_1 = Ax^2 + By^2 + Cz^2 \equiv 2g \pmod{2^n}$, $2 \nmid ABC$, is solvable, with two of x, y, z odd, for arbitrary g and n if the relation $A \equiv B \equiv C \pmod{4}$ does not hold.

Proof: By Theorem 35, stated below, $F_1 \equiv 2g \pmod{2^n}$ is solvable with two of x, y, z odd if $F_1 \equiv 2g \pmod{8}$ is solvable with two of x, y, z odd. The possible even residues modulo 8, with two of x, y, z odd, are $A+B, B+C, C+A, A+B+4, B+C+4, C+A+4$. Now this set of residues contains all even residues modulo 8 if $A \equiv B \equiv C \pmod{4}$ does not hold.

Theorem 35.¹ If $f = ax^2 + by^2 + cz^2 \equiv 2^r k \pmod{8}$ is solvable when a, b, c are odd, with two of x, y, z odd, then $f \equiv 2^r k \pmod{2^n}$ is solvable.

We are now able to prove

Theorem 36. If F is null and three of A, B, C, D are odd and not congruent modulo 4, the remaining coefficient divisible by 4, then there exist solutions of $F=0$ such that $2 \nmid \pi$.

Proof: Assume $2 \nmid ABC$, $A \equiv B \equiv C \pmod{4}$ does not hold, $D = 2^e D'$, $2 \nmid D'$, $e > 0$. Let $x_i = 2^K x'$, $y_i = 2^K y'$, $z_i = 2^K z'$, u_i be a proper solution of $F=0$ such that $\pi = 2^K \pi'$, $2 \nmid \pi'$, $K > 0$. Now $e \geq 2K$ since $2^{2K}(Ax'^2 + By'^2 + Cz'^2) + 2^e D' u_i^2 = 0$.

I. $e > 2K$ implies just one of x', y', z' even. Let $2/x', 2/y', z'$. Then $2 \nmid \sigma \tau MNmn, 2/L, 2/R$. (15) and (23) yield

1. B. W. Jones, Representation by Positive Ternary Quadratic Forms, Chicago Thesis, p. 3.

$$\eta + \zeta \equiv k, \xi \equiv \xi_1 + \lambda T + P, \eta \equiv \eta_1 + T, \zeta \equiv \zeta_1 + T, \phi \equiv \phi_1 + rT + Q \pmod{2}.$$

1) Take $k \equiv k', 2 \nmid k'$. T may be chosen modulo 2 such that η is either odd or even while simultaneously ζ is even or odd. In addition P may be chosen modulo 2 such that $2 \nmid \xi$.

2) Take $k \equiv 2k', 2 \nmid k'$. Select T modulo 2 such that $2 \nmid \eta$. Then $2 \nmid \zeta$. In addition P may be chosen modulo 2 such that $2 \nmid \xi$.

The earlier choices of T, P, k may be made such that $w \equiv 0 \pmod{8}$. On applying Theorem 34 we see that $\xi, \eta, \zeta, \phi, k$, with two of ξ, η, ζ odd, may be chosen satisfying (15) and such that $w \equiv 0 \pmod{2^\theta}$, 2^θ the highest power of 2 contained in $2^{K+1}k$.

II. $e = 2K$.

1) $2 \nmid x'y'z'$ implies $2 \nmid \sigma \tau \text{ LMNn}, 2/R$. (15) and (23) yield

$$\xi + \eta + \zeta \equiv k, \xi \equiv \xi_1 + \lambda T + P, \eta \equiv \eta_1 + mT + P, \zeta \equiv \zeta_1 + T, \\ \phi \equiv \phi_1 + rT + Q \pmod{2}.$$

Take $k \equiv 2k', 2 \nmid k'$. Then T, P, Q may be chosen modulo 2 such that any two of ξ, η, ζ are odd, ϕ even. Hence T, P, Q may be chosen such that $w \equiv 0 \pmod{8}$. On applying Theorem 34 we see that T, P, Q, k may be chosen such that we obtain ξ, η, ζ, ϕ , with two of ξ, η, ζ odd, satisfying (15) and such that $w \equiv 0 \pmod{2^\theta}$, 2^θ the highest power of 2 contained in $2^{K+1}k$.

2) $2 \nmid x', 2 \nmid y', 2 \nmid z'$. Assume $x'y' \neq 0$ since such solutions may be constructed. Let $x' = 2^\alpha x'', y' = 2^\beta y'', 2 \nmid x'', \beta \geq \alpha \geq 1$. Then $2^\alpha/\sigma, 2 \nmid \tau \text{ LMNn}, 2/R$. (15) and (23) yield

$\xi \equiv \xi_1 + \lambda T + MP$, $\eta \equiv \eta_1 + \mu T + P$, $\zeta \equiv \zeta_1 \equiv k$, $\phi \equiv \phi_1 + Q \pmod{2}$.
 $2 \nmid \zeta$ or $2 \mid \zeta$ according as $k \equiv k'$ or $k \equiv 2k'$, $2 \nmid k'$. ξ, η can be made to take any desired residues modulo 2 by choice of T , P modulo 2 since $L\lambda + M\mu = 1$. In addition Q may be chosen modulo 2 such that $2 \nmid \phi$. Hence T, P, Q, k may be chosen such that two of ξ, η, ζ are odd and $w \equiv 0 \pmod{8}$. On applying Theorem 34 we see that T, P, Q, k may be chosen such that two of ξ, η, ζ are odd and $w \equiv 0 \pmod{2^\theta}$, 2^θ the highest power of 2 contained in $2^{K+1}k$.

Therefore, in any case, we can construct solutions
 (31) $X = 2^{K+1} \pi' k \xi - w x_1$, $Y = 2^{K+1} \pi' k \eta - w y_1$, $Z = 2^{K+1} \pi' k \zeta - w z_1$,
 $U = 2^{K+1} \pi' k \phi - w u_1$ of $F = 0$. From (31) we obtain solutions
 (32) $X_1 = \frac{1}{2^\theta} X$, $Y_1 = \frac{1}{2^\theta} Y$, $Z_1 = \frac{1}{2^\theta} Z$, $U_1 = \frac{1}{2^\theta} U$, 2^θ the highest power of 2 contained in $2^{K+1}k$, where two of X_1, Y_1, Z_1 are odd.

Theorem 37. Every null F satisfying (12) and (13) and having three of A, B, C, D odd and not congruent modulo 4, the remaining coefficient even, is universal.

Proof: Assume $2 \nmid ABC$, $A \equiv B \equiv C \pmod{4}$ does not hold, $D = 2^e D'$, $2 \nmid D'$, $e > 0$. Theorem 33 disposes of $D \equiv 2 \pmod{4}$.

In case $4 \mid D$ Theorem 36 declares the existence of solutions of $F = 0$ such that $2 \nmid \pi$ while Theorem 31 states there exist solutions such that π contains no odd prime factors p if p^2 divides any one of the coefficients of F . Hence there exist solutions such that $2 \nmid \pi$ and π contains no odd prime factors p , p^2 a divisor of any one of the coefficients of F . By Theorem 18, $w \equiv g \pmod{\pi}$ is solvable for arbitrary g

since π is now 1 or a product of distinct odd primes p , p^2 not dividing any one of A , B , C , D . In order to complete the proof we must show $w \equiv g \pmod{2k\pi}$ solvable for arbitrary g . Since $2 \nmid \pi$, we may suppose $2/x_1, 2/y_1, z_1$. Then $2/L, 2/\sigma \tau MNmn, 2/R$. Take $k=1$. (15) and (18) yield $w \equiv \xi + \eta + \zeta \equiv \xi + 1 \pmod{2}$. In (23) we see that $\xi \equiv \xi_1 + \lambda T + P \pmod{2}$. Hence, by choice of T , P , modulo 2, it is possible to obtain $w \equiv 1$ or $0 \pmod{2}$. From this we conclude $w \equiv g \pmod{2\pi}$ solvable for arbitrary g .

Theorem 7 declares F is not universal when three of A , B , C , D are congruent modulo 4, the fourth $\equiv 0 \pmod{8}$. It remains to consider three of A , B , C , D odd and congruent modulo 4, the fourth $\equiv 4 \pmod{8}$.

Theorem 38. Every null F satisfying (12) and (13) and having three of A , B , C , D odd and congruent modulo 4, the fourth $\equiv 4 \pmod{8}$, is universal.

Proof: Let $2 \nmid ABC$, $A \equiv B \equiv C \pmod{4}$, $D = 4D'$, $2 \nmid D'$. Then $\pi = 2\pi'$, $2 \nmid \pi'$. Write $x_1 = 2x'$, $y_1 = 2y'$, $z_1 = 2z'$, where x_1, y_1, z_1, u_1 is a proper solution of $F = 0$. (15), (17) and (18) yield

$$(33) \quad Ax'\xi + By'\eta + Cz'\zeta + 2D'u_1\phi = \pi'k,$$

$$(34) \quad 4\pi'kh = g - w,$$

$$(35) \quad w = A\xi^2 + B\eta^2 + C\zeta^2 + 4D'\phi^2.$$

I. $2 \nmid x'y'z'u_1$ implies $2/\sigma \tau LMNn, 2/R$. (23), (33) yield

$$\xi \equiv \xi_1 + \lambda T + P, \quad \eta \equiv \eta_1 + mT + P, \quad \zeta \equiv \zeta_1 + T, \quad \phi \equiv \phi_1 + rT + Q,$$

$$\xi + \eta + \zeta \equiv k \pmod{2}.$$

1) Take $k=1$. Then $w \equiv \xi + \eta + \zeta \equiv 1 \pmod{2}$.

a) We may take T, P modulo 2 such that $2/\eta, 2/\zeta$. Then $2/\xi$ and $w \equiv A \pmod{4}$.

b) We may take T, P modulo 2 such that $2/\eta\zeta$. Then $2/\xi$ and $w \equiv A+2 \pmod{4}$.

2) Take $k=2$. Then $w \equiv \xi + \eta + \zeta \equiv 0 \pmod{2}$.

a) We may take T, P modulo 2 such that $2/\xi\eta$. Then $2/\zeta$ and $w \equiv A+B+C\xi^2 + 4D'\phi^2 \pmod{8}$. In addition Q may be chosen modulo 2 such that ϕ is odd or even as desired. Hence we are able to obtain $w \equiv A+B$ or $w \equiv A+B+4 \pmod{8}$ at will.

b) We may take T, P modulo 2 such that $2/\xi, 2/\eta$. Then $2/\zeta$. Again Q may be chosen modulo 2 such that ϕ is odd or even as desired. Hence we are able to obtain $w \equiv 4$ or $w \equiv 0 \pmod{8}$ at will.

II. $2/x', 2/y', 2/z'$ imply $2/\sigma LR\bar{z}, 2/M, \tau \equiv 2 \pmod{4}$.

(23) and (33) yield

$$\xi \equiv \xi_i \equiv k, \eta \equiv \eta_i + P, \zeta \equiv \zeta_i + nT + Q, \phi \equiv \phi_i + rT + nQ \pmod{2}.$$

1) Take $k=1$. Then $2/\xi$.

a) We may take T, P, Q modulo 2 such that $2/\eta, 2/\zeta$. Then $w \equiv A \pmod{4}$.

b) We may take T, P, Q modulo 2 such that $2/\eta\zeta$. Then $w \equiv A+2 \pmod{4}$.

c) We may take T, P, Q modulo 2 such that $2/\eta, 2/\zeta$. Then $w \equiv 2 \pmod{4}$.

2) Take $k=2$. Then $2/\xi$.

a) We may take T, P, Q modulo 4 such that $\xi \equiv 2$,

$\eta \equiv \zeta \equiv 0$ or $2 \pmod{4}$. Then $2/\phi$ and $w \equiv 4 \pmod{8}$.

b) We may take T, P, Q modulo 4 such that

$\xi \equiv \eta \equiv 2, \zeta \equiv 0 \pmod{4}$. Then $2/\phi$ and $8/w$.

In view of Theorem 31, we may assume π' is either 1 or a product of distinct odd primes p , p^2 dividing no one of A, B, C, D . Then, according to Theorem 18, $w \equiv g \pmod{\pi'}$ is solvable for arbitrary g . From this and the above we conclude $w \equiv g \pmod{2k\pi}$, $k=1$ or 2 as necessary, is solvable for arbitrary g .

Theorem 39. Every null F having at least three odd coefficients and satisfying (6), (12) and (15) is universal.

Proof: This follows as a consequence of Theorems 33, 37 and 38.

6.2. JUST TWO OF A, B, C, D EVEN.

In the sequel we apply

Theorem 40. $Ax^2 + By^2 \equiv 0 \pmod{2^n}$ is solvable for arbitrary n , with $2 \nmid xy$, if $A+B \equiv 0 \pmod{8}$, $2 \nmid AB$.

Proof: There are solutions $x=\xi$, $y=\eta$, $2 \nmid \xi\eta$ when $n=3$. To proceed by induction on n , let $Ax^2 + By^2 \equiv 0 \pmod{2^m}$, $m \geq 3$, have a solution ξ, η , $2 \nmid \xi\eta$. Write $A\xi^2 + B\eta^2 = 2^m q$, $x = \xi + 2^{m-1}X$, $y = \eta + 2^{m-1}Y$, $2 \nmid \xi\eta$. Then

$$\begin{aligned} Ax^2 + By^2 &= A\xi^2 + B\eta^2 + 2^m(A\xi X + B\eta Y) + 2^{2m-2}(AX^2 + BY^2) \\ &\equiv 2^m(A\xi X + B\eta Y + q) \pmod{2^{m+1}}. \end{aligned}$$

Since $2 \nmid AB\xi\eta$, there exist X, Y such that $A\xi X + B\eta Y + q \equiv 0 \pmod{2}$. Then $Ax^2 + By^2 \equiv 0 \pmod{2^{m+1}}$ and the induction is complete.

Theorem 41. $Ax^2 + By^2 + Cz^2 \equiv 0 \pmod{2^n}$ is solvable for

arbitrary n , with $2 \nmid xyz$, if $A+B+C \equiv 0 \pmod{8}$, $2 \nmid AB$.

The method of proof is similar to that of Theorem 40 and consequently not given.

In this section we study $F = Ax^2 + By^2 + 2C'z^2 + 2^e D'u^2$, $2 \nmid ABC'D'$, $e > 1$. Theorem 33 disposes of $e = 1$.

6.21. $A+B \equiv 0 \pmod{8}$.

Theorem 42. There exist solutions of $F = 0$ such that $2 \nmid \pi$ if F is null, exactly two of the coefficients of F are even, but not both divisible by 4, and the sum of the odd coefficients $\equiv 0 \pmod{8}$.

Proof: Let x_1, y_1, z_1, u_1 be a proper solution of $F \equiv Ax^2 + By^2 + 2C'z^2 + 2^e D'u^2 = 0$, $2 \nmid ABC'D'$, $A+B \equiv 0 \pmod{8}$, $e > 0$, such that $\pi = 2^K \pi'$, $2 \nmid \pi'$, $K \geq 1$. Write $x_1 = 2^K x'$, $y_1 = 2^K y'$, $z_1 = 2^{K-1} z'$. We may assume $x, y_1 \neq 0$ since Theorem 15 shows such solutions exist. Now $e \geq 2K-1$.

I. $e = 2K-1$ implies $2 \nmid z'u_1 \in \mathbb{N}$.

1) $2 \nmid x'$, $2 \nmid y'$ imply $2 \nmid \sigma LY$, $2/M$. (23) and (15)

yield

$$\begin{aligned} \xi &\equiv \xi_1 + T, \quad \eta \equiv \eta_1 + mT + P, \quad \zeta \equiv \zeta_1 + nT + RQ, \quad \phi \equiv \phi_1 + rT + Q, \\ \xi + \zeta + 2^{K-1} \phi &\equiv k \pmod{2}. \end{aligned}$$

Take $k \equiv k'$, $2 \nmid k'$. We may take T, P, Q modulo 2 such that $2 \nmid \xi\eta$, $2 \nmid \phi$. Then $2 \nmid \zeta$ and $8 \nmid w$. Now, according to Theorem 40, we may take T, P, Q, k , $2 \nmid k$, in addition, such that $2 \nmid \xi\eta$, $2 \nmid \zeta$, $2 \nmid \phi$ and $2^{e+1} \nmid w$. Hence, from (24), we are able to obtain solutions $X_1 = \frac{1}{2^{K+1}} X$, $Y_1 = \frac{1}{2^{K+1}} Y$, $Z_1 = \frac{1}{2^{K+1}} Z$, $U_1 = \frac{1}{2^{K+1}} U$ of $F = 0$ with $2 \nmid X, Y_1$.

2) $2 \nmid x'y'$ implies $2 \nmid \sigma LM$. (23) and (15) yield

$$\xi \equiv \xi_1 + \chi T + P, \quad \eta \equiv \eta_1 + mT + P, \quad \zeta \equiv \zeta_1 + nT + RQ, \quad \phi \equiv \phi_1 + rT + Q, \\ \xi + \eta + \zeta + 2^{K-1}\phi \equiv k \pmod{2}.$$

Take $k = 2k'$, $2 \nmid k'$. We may take T, P, Q modulo 2 such that $2 \nmid \xi\eta$, $2/\phi$. Then $2/\zeta$ and $8/w$. In addition, according to Theorem 40, we may take $T, P, Q, k', 2 \nmid k'$, such that $2 \nmid \xi\eta$, $2/\zeta$, $2/\phi$ and $2^{e+2}/w$. Hence, from (24) we are able to obtain solutions $X_1 = \frac{1}{2^{K+2}} X$, $Y_1 = \frac{1}{2^{K+2}} Y$, $Z_1 = \frac{1}{2^{K+2}} Z$, $U_1 = \frac{1}{2^{K+2}} U$ of $F=0$ with $2 \nmid X, Y_1$.

3) $2/x', 2/y'$. Write $x' = 2^\alpha x''$, $y' = 2^\beta y''$, $\beta \geq \alpha \geq 1$, $2 \nmid x''y''$. Then $2^\alpha/\sigma$, $2 \nmid L$. (23) and (15) yield

$$\xi \equiv \xi_1 + \chi T + MP, \quad \eta \equiv \eta_1 + mT + P, \quad \zeta \equiv \zeta_1 + RQ, \quad \phi \equiv \phi_1 + Q, \\ \zeta + 2^{K-1}\phi \equiv k \pmod{2}.$$

Take $k = 2k'$, $2 \nmid k'$. We may take T, P, Q modulo 2 such that $2 \nmid \xi\eta$, $2/\zeta$. Then $8/w$. From (24), we obtain solutions $X_1 = \frac{1}{8} X$, $Y_1 = \frac{1}{8} Y$, $Z_1 = \frac{1}{8} Z$, $U_1 = \frac{1}{8} U$ of $F=0$ such that the corresponding $2K-1 < e$.

This leaves for our consideration

II. $e \geq 2K$ implies $2/z', 2 \nmid u_1$. Let $y' = 2^\beta y''$, $z' = 2^\gamma z''$, $2 \nmid x'y''$. Then $2 \nmid \sigma L$, $2/\tau$.

1) $\beta > 0$ implies $e = 2K$, $2 \nmid \chi$, $2/M$. (15) and (23) yield

$$\xi \equiv \xi_1 \equiv k, \quad \eta \equiv \eta_1 + P, \quad \zeta \equiv \zeta_1 + nT + RQ, \quad \phi \equiv \phi_1 + rT + NQ \pmod{2}.$$

Take $k = k'$, $2 \nmid k'$. Then $2 \nmid \xi$. We may take T, P, Q modulo 2 such that $2 \nmid \eta$, $2/\zeta$, $2/\phi$. Then $8/w$. In addition, according to Theorem 40, we may take $T, P, Q, k, 2 \nmid k$, such that $2 \nmid \xi\eta$, $2^K/\zeta$, $2/\phi$ and $2^{2K+1}/w$.

2) $\beta = 0$ implies $e > 2K$, $2 \nmid M$. (15) and (23) yield

$\xi + \eta \equiv k$, $\xi \equiv \xi_1 + P$, $\eta \equiv \eta_1 + P$, $\zeta \equiv \zeta_1 + nT + RQ$, $\phi \equiv \phi_1 + rT + NQ \pmod{2}$.
 Take $k = 2k'$, $2 \nmid k'$. We may take T, P, Q modulo 2 such that
 $2 \nmid \xi$, $2 \nmid \zeta$. Then $2 \nmid \eta$ and $8 \nmid w$. In addition, according to
 Theorem 40, we may take $T, P, Q, k', 2 \nmid k'$, such that $2 \nmid \xi \eta$,
 $2^K \nmid \zeta$ and $2^{2K+1} \nmid w$.

From (24), in either case we obtain solutions $X_1 = \frac{1}{2^\theta} X$,
 $Y_1 = \frac{1}{2^\theta} Y$, $Z_1 = \frac{1}{2^\theta} Z$, $U_1 = \frac{1}{2^\theta} U$, $2\pi k = 2^\theta \pi' k'$, $2 \nmid \pi' k'$, of
 $F = 0$ with $2 \nmid X, Y_1$.

Theorem 43. Every null F satisfying (5), (12), (13) and having exactly two of A, B, C, D even and the sum of the odd coefficients $\equiv 0 \pmod{8}$ is universal.

Proof: Theorem 33 disposes of cases when no one of A, B, C, D divisible by 4. Theorems 42 and 31 assert the existence of solutions of $F = 0$ such that π is odd and contains no prime factors p , p^2 a divisor of any one of the coefficients of F . By Theorem 18, $w \equiv g \pmod{\pi}$ is solvable for arbitrary g when π is a product of distinct odd primes p , p^2 not dividing any one of A, B, C, D . It remains to determine T, P, Q, k modulo 2 or 4 such that $w \equiv g \pmod{2\pi k}$ is solvable. Let x_1, y_1, z_1, u_1 be a proper solution of $F \equiv Ax^2 + By^2 + 2C'z^2 + 2^e D'u^2 = 0$, $2 \nmid ABC'D'$, $A + B \equiv 0 \pmod{8}$, $e > 0$ such that π is odd and contains no prime factors p if p^2 divides any one of A, B, C, D . Then $2 \nmid \sigma LM$, $2 \nmid \tau$. (15) and (23) yield

$$\xi + \eta \equiv k, \xi \equiv \xi_1 + P, \eta \equiv \eta_1 + P, \zeta \equiv \zeta_1 + nT + RQ, \phi \equiv \phi_1 + rT + NQ \pmod{2}.$$

1) $k = 1$ implies $w \equiv 1 \pmod{2}$.

2) $k = 2$.

a) We may take T, P, Q modulo 2 such that $2 \nmid \eta$,

$2/\xi$, $2/\phi$. Then $2/\xi$ and $w \equiv 2 \pmod{4}$.

b) We may take T, P, Q modulo 2 such that $2/\eta$, $2/\xi$, $2/\phi$. Then $2/\xi$ and $w \equiv 0 \pmod{4}$.

6.22. $A+B \equiv 4 \pmod{8}$.

From Theorem 11 we learn that no $F \equiv Ax^2 + By^2 + 2C'z^2 + 2^e D'u^2$, $2 \nmid ABC'D'$, $A+B \equiv 4 \pmod{8}$, $e \geq 4$, is universal. Having disposed of $e=1$ in Theorem 33 it remains for us to consider $e=2$ and $e=3$.

Theorem 44. If F is null there exist solutions of $F=0$ such that $2 \nmid \pi$ when one of A, B, C, D is $\equiv 2 \pmod{4}$, a second $\equiv 4 \pmod{8}$, the remaining coefficients odd and such that their sum is $\equiv 4 \pmod{8}$.

Proof: Let x_1, y_1, z_1, u_1 be a proper solution of $F \equiv Ax^2 + By^2 + 2C'z^2 + 4D'u^2 = 0$, $2 \nmid ABC'D'$, $A+B \equiv 4 \pmod{8}$ such that $\pi = 2\pi'$, $2 \nmid \pi'$. Without loss of generality we may write $x_1 = 2x'$, $y_1 = 4y'$, $z_1 = 2z'$, $2 \nmid x'u_1$. Then $2 \nmid \sigma L R X$, $2/M$, $2/\tau$. (15) and (23) yield

$\xi \equiv \xi_1 \equiv k$, $\eta \equiv \eta_1 + P$, $\zeta \equiv \zeta_1 + nT + Q$, $\phi \equiv \phi_1 + rT + nQ \pmod{2}$. Take $k = k'$, $2 \nmid k'$. Then $2 \nmid \xi$. We may take T, P, Q modulo 2 such that $2 \nmid \eta \phi$, $2/\xi$. Then $8/w$. From (24) we obtain solutions $X_1 = \frac{1}{4} X$, $Y_1 = \frac{1}{4} Y$, $Z_1 = \frac{1}{4} Z$, $U_1 = \frac{1}{4} U$ of $F=0$ with $2 \nmid X, Y_1$.

Theorem 45. Every null F satisfying (12) and (13) is universal when one of A, B, C, D is $\equiv 2 \pmod{4}$, a second $\equiv 4 \pmod{8}$, the remaining coefficients odd and such that their sum is $\equiv 4 \pmod{8}$.

Proof: Theorems 31 and 44 assert the existence of solutions of $F=0$ such that π is odd and contains no prime factors

p, p^2 a divisor of any one of the coefficients of F . By Theorem 18, $w \equiv g \pmod{\pi}$ is solvable for arbitrary g when π is a product of distinct odd primes p, p^2 not dividing any one of A, B, C, D . It remains to determine T, P, Q, k modulo 2 or 4 such that $w \equiv g \pmod{2\pi k}$ is solvable. Let x_1, y_1, z_1, u_1 be a proper solution of $F \equiv Ax^2 + By^2 + 2C'z^2 + 4D'u^2 = 0$, $2 \nmid ABC'D'$, $A+B \equiv 4 \pmod{8}$, such that π is odd and contains no prime factors p if p^2 divides any one of A, B, C, D . Then $2 \nmid x_1, y_1, u_1, 2 \nmid z_1, 2 \nmid \sigma \text{ LMR}, 2 \nmid \tau$. (15) and (23) yield

$$\xi + \eta \equiv k, \xi \equiv \xi_1 + P, \eta \equiv \eta_1 + P, \zeta \equiv \zeta_1 + nT + Q, \phi \equiv \phi_1 + rT + nQ \pmod{2}.$$

1) $k=1$ implies $w \equiv 1 \pmod{2}$.

2) $k=2$. We obtain $w \equiv 2$ or $0 \pmod{4}$ according as T, Q are chosen such that $2 \nmid \zeta$ or $2 \mid \zeta$.

Theorem 46. Every null F satisfying (12) and (13) is universal if one of A, B, C, D is $\equiv 2 \pmod{4}$, a second $\equiv 8 \pmod{16}$, the remaining coefficients odd and such that their sum is $\equiv 4 \pmod{8}$.

Proof: Theorem 31 asserts the existence of solutions of $F=0$ such that π contains no odd prime factor p, p^2 a divisor of any one of the coefficients of F . Let x_1, y_1, z_1, u_1 be a proper solution of $F \equiv Ax^2 + By^2 + 2C'z^2 + 8D'u^2 = 0$, $2 \nmid ABC'D'$, $A+B \equiv 4 \pmod{8}$, such that $\pi = 2^K \pi'$, π' odd and containing no prime factors p if p^2 divides any one of A, B, C, D . Then, according to Theorem 18, $w \equiv g \pmod{\pi'}$ is solvable for arbitrary g . It remains to determine T, P, Q, k modulo 2, 4, 8 such that $w \equiv g \pmod{2\pi k}$ is solvable for arbitrary g .

I. $C' + D' \equiv 2 \pmod{4}$

1) $K=2$. We may write $x_1=4x'$, $y_1=8y'$, $z_1=2z'$,
 $\pi=4\pi'$, $2/x'z'u, \pi'$. Then $2/\sigma\tau LN/n$, $2/M$, $R \equiv 2 \pmod{4}$.
 (15) and (23) yield

$\xi+\eta \equiv k$, $\xi \equiv \xi_1+T$, $\eta \equiv \eta_1+mT+P$, $\zeta \equiv \zeta_1+T$, $\phi \equiv \phi_1+rT+Q \pmod{2}$.
 Take $k=2k'$, $2/\phi$, and T, P, Q modulo 4 such that $\xi \equiv \eta \equiv \zeta \equiv 2 \pmod{4}$.
 Then $2/\phi$ and $32/w$. Hence $X_1=\frac{1}{16}X$, $Y_1=\frac{1}{16}Y$,
 $Z_1=\frac{1}{16}Z$, $U_1=\frac{1}{16}U$, are solutions of $F=0$ with $X_1 \equiv Y_1 \equiv Z_1 \equiv 2 \pmod{4}$,
 $2/U_1$, where X, Y, Z, U are defined by (24). This leads to a study of the case $K=1$.

2) $K=1$. We may write $x_1=2x'$, $y_1=2y'$, $z_1=2z'$,
 $\pi=2\pi'$, $2/x'y'z'u, \pi'$. Then $2/\sigma LMNn$, $\tau \equiv R \equiv 2 \pmod{4}$. (15)
 and (23) yield

$\xi+\eta \equiv k$, $\xi \equiv \xi_1+P$, $\eta \equiv \eta_1+P$, $\zeta \equiv \zeta_1+T$, $\phi \equiv \phi_1+rT+Q \pmod{2}$.

a) Take $k=1$. We may take T, P modulo 2 such that $2/\xi$ and $2/\zeta$ or $2/\zeta$. Then $2/\eta$ and $w \equiv A+2$ or $A \pmod{4}$ according as $2/\xi$ or $2/\zeta$.

b) Take $k=2$.

1) We may take T, P modulo 2 such that $2/\xi\zeta$. Then $2/\eta$ and $w \equiv 2C'+4 \pmod{8}$.

2) We may take T, P modulo 4 such that $\xi \equiv \eta \equiv 2$ or $0 \pmod{4}$. Then $2/\zeta$ and $w \equiv 2C' \pmod{8}$.

3) We may take T, P, Q modulo 2 such that $2/\eta$, $2/\zeta$. Then $2/\xi$ and $w \equiv 4 \pmod{8}$.

c) Take $k=4$. We may take T, P, Q modulo 4 such that $\eta \equiv 2$, $\zeta \equiv 0 \pmod{4}$, $2/\phi$ or $2/\phi$. Then $\xi \equiv 2 \pmod{4}$ and $w \equiv 8$ or $0 \pmod{16}$ according as $2/\phi$ or $2/\phi$.

II. $C'+D' \equiv 0 \pmod{8}$. $K=2$ always. Write $x_1=4x'$,

$y_1 = 4y'$, $z_1 = 2z'$, $\pi = 4\pi'$, $2/z'u, \pi'$. Then $2/\tau Nn$, $R \equiv 2 \pmod{4}$.

1) $2/x'$ implies $2/y'$, $2/\sigma$. We may assume $x'y' \neq 0$ since such solutions exist. Take $k=k'$, $2/k'$. Then $2/\xi$. We may take T, P modulo 2 such that $2/\xi\eta$ since $2/\tau$ and $L\chi + Mm = 1$. Then $w \equiv 2 \pmod{4}$ and $X_1 = \frac{1}{2}X$, $Y_1 = \frac{1}{2}Y$, $Z_1 = \frac{1}{2}Z$, $U_1 = \frac{1}{2}U$ are solutions of $F=0$ with $X_1 \equiv Y_1 \equiv 4 \pmod{8}$, $Z_1 \equiv 2 \pmod{4}$, $2/U_1$, where X, Y, Z, U are defined by (24). Hence we study:

2) $2/x'y'$ implies $2/\sigma LM$. (15) and (23) yield

$$\begin{aligned} \xi + \eta + \zeta &\equiv k, \quad \xi \equiv \xi_1 + \chi T + P, \quad \eta \equiv \eta_1 + mT + P, \quad \zeta \equiv \zeta_1 + T, \\ \phi &\equiv \phi_1 + rT + Q \pmod{2}. \end{aligned}$$

a) Take $k=1$.

1) We may take T, P modulo 4 such that $2/\xi$, $\eta \equiv 0$ or $2 \pmod{4}$. Then $2/\xi$ and $w \equiv A$ or $A+4 \pmod{8}$ in agreement with choice of η .

2) We may take T, P modulo 4 such that $\xi \equiv 0$ or $2 \pmod{4}$, $2/\eta$. Then $2/\xi$ and $w \equiv B$ or $B+4 \pmod{8}$ in agreement with choice of ξ .

3) We may take T, P modulo 4 such that $\xi \equiv \eta \equiv 2$ or $0 \pmod{4}$. Then $2/\xi$ and $w \equiv 2C' \pmod{8}$.

4) We may take T, P modulo 2 such that $2/\xi\eta$. Then $2/\xi$ and $w \equiv 2C' + 4 \pmod{8}$.

b) Take $k=2$.

1) We may take T, P, Q modulo 8 such that $\xi \equiv x' + 4$, $\eta \equiv y' \pmod{8}$, $2/\phi$. Then $4/\xi$ and $w \equiv Ax'^2 + By'^2 \pmod{16}$.

2) We may take T, P, Q modulo 8 such that $\xi \equiv x', \eta \equiv y' \pmod{8}$, $2 \nmid \phi$. Then $4/\xi$ and $w \equiv Ax'^2 + By'^2 + 8 \pmod{16}$. 3) We may take T, P, Q modulo 4 such that

$\xi \equiv \eta \equiv 2$ or $0 \pmod{4}$, $4/\xi$. Then $2 \nmid \phi$ and $w \equiv 8 \pmod{16}$.

c) Take $k=4$.

1) We may take T, P, Q modulo 4 such that $\xi \equiv \eta \equiv 2 \pmod{4}$, $4/\xi$. Then $2/\phi$ and $w \equiv 16 \pmod{32}$.

2) We may take T, P, Q modulo 8 such that $\xi \equiv \eta \equiv 4$ or $0 \pmod{8}$, $4/\xi$. Then $2/\phi$ and $32/w$.

6.23. $A+B \equiv 2 \pmod{4}$.

For the purposes of this section we may assume, without loss of generality, that $C=2C'$, $D=2^e D'$, $2 \nmid ABC'D'$. Then $F \equiv Ax^2 + By^2 + 2C'z^2 + 2^e D'u^2$.

6.231. $A+B+2C' \equiv 0 \pmod{8}$.

Theorem 47. If $F \equiv Ax^2 + By^2 + 2C'z^2 + 2^e D'u^2$, $2 \nmid ABC'D'$, is null and $A+B+2C' \equiv 0 \pmod{8}$, there exist solutions of $F=0$ such that $2 \nmid \pi$.

Proof: $2 \nmid \pi$ when $e=0$. Now, let $x_i = 2^K x'$, $y_i = 2^K y'$, $z_i = 2^{K-1} z'$, u_i be a proper solution of $F=0$ such that $\pi = 2^K \pi'$, $2 \nmid \pi'$, $K \geq 1$. Then $e \geq 2K-1$.

I. $e=2K-1$ implies $2 \nmid z' \nmid \tau Nu_i$.

1) $2 \nmid x'y'$ implies $2 \nmid \sigma LM$. (15) and (23) yield

$$\xi + \eta + \zeta + 2^{K-1} \phi \equiv k, \quad \xi \equiv \xi_i + \lambda T + P, \quad \eta \equiv \eta_i + m T + P,$$

$$\zeta \equiv \zeta_i + n T + R Q, \quad \phi \equiv \phi_i + r T + Q \pmod{2}.$$

Take $k=k'$, $2 \nmid k'$. We may take T, P, Q modulo 2 such that $2 \nmid \xi \eta$, $2/\phi$. Then $2 \nmid \xi$ and $8/w$. Now, according to Theorem 41, we may take T, P, Q, k' , $2 \nmid k'$, such that $2 \nmid \xi \eta \zeta$, $2/\phi$

and $2^{2K+1}/w$.

2) $2/x', 2/y'$ imply $2/\sigma L$, $2/M$. (15) and (23)

yield

$$\begin{aligned}\xi + \zeta + 2^{K-1}\phi &\equiv k, \quad \xi \equiv \xi_1 + T, \quad \eta \equiv \eta_1 + mT + P, \quad \zeta \equiv \zeta_1 + nT + RQ, \\ \phi &\equiv \phi_1 + rT + Q \pmod{2}.\end{aligned}$$

Take $k=2k'$, $2/k'$. We may take T, P, Q modulo 2 such that $2/\xi\eta$, $2/\phi$. Then $2/\zeta$ and $8/w$. Now, according to Theorem 41, we may take $T, P, Q, k', 2/k'$, such that $2/\xi\eta\zeta$, $2/\phi$ and $2^{2K+1}/w$.

3) $2/x', 2/y'$ imply $2/\sigma$. We may assume $x' \neq 0$.

(15) and (23) yield

$$\begin{aligned}\zeta + 2^{K-1}\phi &\equiv k, \quad \xi \equiv \xi_1 + \lambda T + MP, \quad \eta \equiv \eta_1 + mT + LP, \quad \zeta \equiv \zeta_1 + RQ, \\ \phi &\equiv \phi_1 + Q \pmod{2}.\end{aligned}$$

Take $k=k'$, $2/k'$. We may take T, P, Q modulo 2 such that $2/\xi\eta$, $2/\phi$. Then $2/\zeta$ and $8/w$. Now, according to Theorem 41, we may take $T, P, Q, k', 2/k'$, such that $2/\xi\eta\zeta$, $2/\phi$ and $2^{2K+1}/w$.

II. $e \geq 2K$ implies $2/z', 2/\tau$. We may assume $2/x'$, $y' = 2^\beta y''$, $2/y''$. Then $2/\sigma L$.

1) $\beta > 0$ implies $e=2K$, $2/\lambda$, $2/M$. (15) and (23)

yield

$$\xi \equiv k, \quad \eta \equiv \eta_1 + P, \quad \zeta \equiv \zeta_1 + nT + RQ, \quad \phi \equiv \phi_1 + rT + NQ \pmod{2}.$$

Take $k=k'$, $2/k'$. Then $2/\xi$. We may take T, P, Q modulo 2 such that $2/\eta\zeta$, $2/\phi$. Then $8/w$. Now, according to Theorem 41, we may take $T, P, Q, k', 2/k'$ such that $2/\xi\eta\zeta$, $2/\phi$ and $2^{2K+2}/w$.

2) $\beta = 0$ implies $e > 2K$, $2/M$. (15) and (23) yield

$\xi + \eta \equiv k, \xi \equiv \xi_i + P, \eta \equiv \eta_i + P, \zeta \equiv \zeta_i + nT + RQ, \phi \equiv \phi_i + rT + NQ \pmod{2}$.
 Take $k \equiv 2k', 2 \nmid k'$. We may take T, P, Q modulo 2 such that
 $2 \nmid \eta \zeta, 2 \nmid \phi$. Then $2 \nmid \xi$ and $8 \nmid w$. Now, according to Theorem
 41, we may take $T, P, Q, k', 2 \nmid k'$, such that $2 \nmid \xi \eta \zeta, 2 \nmid \phi$
 and $2^{2K+3}/w$.

Therefore, in any case when $e > 0$, we are able to construct solutions $X_i = \frac{1}{2^e} X, Y_i = \frac{1}{2^e} Y, Z_i = \frac{1}{2^e} Z, U_i = \frac{1}{2^e} U$ of $F=0$ with $2 \nmid X, Y$, where X, Y, Z, U are defined by (24) and $2\pi k = 2^e \pi' k', 2 \nmid \pi' k'$.

Theorem 48. Every null $F \equiv Ax^2 + By^2 + 2C'z^2 + 2^e D'u^2, 2 \nmid ABC'D', A+B+2C' \equiv 0 \pmod{8}$, satisfying (12) and (13) is universal.

Proof: Theorem 33 disposes of $e = 0, 1$. Now, let $e \geq 2$. Then, Theorems 31 and 47 assert the existence of solutions of $F=0$ such that π is odd and contains no prime factors p if p^2 divides any one of the coefficients of F . By Theorem 18, $w \equiv g \pmod{\pi}$ is solvable for arbitrary g when π is a product of distinct odd primes p, p^2 not dividing any one of the coefficients of F . It remains to determine T, P, Q, k modulo 2 or 4 such that $w \equiv g \pmod{2\pi k}$ is solvable for arbitrary g . Let x_i, y_i, z_i, u_i be a proper solution of $F=0$ such that π is odd and contains no prime factors p if p^2 divides any one of the coefficients of F . Then $2 \nmid \sigma LMNn, 2 \nmid \tau, 2 \nmid R$. (15) and (23) yield

$$\xi + \eta \equiv k, \xi \equiv \xi_i + P, \eta \equiv \eta_i + P, \zeta \equiv \zeta_i + T, \phi \equiv \phi_i + rT + Q \pmod{2}.$$

1) Take $k=1$. Then $w \equiv 1 \pmod{2}$.

2) Take $k=2$. We may take T, P, Q modulo 2 such that

$2/\xi$, $2/\zeta$ or $2/\zeta$, $2/\phi$. Then $2/\eta$ and $w \equiv 2$ or $0 \pmod{4}$ according as $2/\xi$ or $2/\zeta$.

6.232. $A+B \equiv 2C' \pmod{8}$.

Theorems 10 and 12 assert there is no universal F having $A+B \equiv 2C'$, $A \equiv B$ or $B+4 \pmod{8}$, $e \geq 4$. Theorem 33 dispose of $e=0, 1$. It remains for us to consider $e=2$ and 3 .

Theorem 49. If $F \equiv Ax^2 + By^2 + 2C'z^2 + 4D'u^2$, $2/\overline{ABC'D'}$, $F \neq 0$ null and $A+B+2C'+4D' \equiv 0 \pmod{8}$, there exist solutions of $F=0$ such that $2/\pi$.

Proof: Let $x_1 = 2x'$, $y_1 = 2y'$, $z_1 = 2z'$, u_1 be a proper solution of $F=0$ such that $\pi = 2\pi'$, $2/\pi'$. We may assume $2/\overline{x'}$, $2/\overline{y'}$. Then $2/\sigma \overline{LRX}$, $2/M$, $2/\tau$. (15) and (23) yield

$$\xi \equiv k, \eta \equiv \eta_1 + P, \zeta \equiv \zeta_1 + nT + Q, \phi \equiv \phi_1 + rT + NQ \pmod{2}.$$

Take $k=1$. Then $2/\xi$. We may take T, P, Q modulo 2 such that $2/\eta\zeta\phi$. Then $8/w$. Therefore $X_1 = \frac{1}{4}X$, $Y_1 = \frac{1}{4}Y$, $Z_1 = \frac{1}{4}Z$, $U_1 = \frac{1}{4}U$ are solutions of $F=0$ with $2/\overline{X_1Y_1Z_1U_1}$, where X, Y, Z, U are defined by (24).

Theorem 50. Every null $F \equiv Ax^2 + By^2 + 2C'z^2 + 4D'u^2$, $2/\overline{ABC'D'}$, $A+B+2C'+4D' \equiv 0 \pmod{8}$, satisfying (12) and (13) is universal.

Proof: Theorems 31 and 49 assert the existence of solutions of $F=0$ such that π is odd and contains no prime factors p if p^2 divides any one of A, B, C', D' . By Theorem 18, $w \equiv g \pmod{\pi}$ is solvable for arbitrary g when π is a product of distinct odd primes p , p^2 not dividing any one of A, B, C', D' . It remains to determine T, P, Q, k modulo 2 or 4 such that $w \equiv g \pmod{2\pi k}$ is solvable for arbitrary g . Let $x_1, y_1,$

z_1, u_1 be a proper solution of $F=0$ such that π is odd and contains no prime factors p if p^2 divides any one of A, B, C', D' . Then $2/\sigma LMRn, 2/\tau, 2/R$. (15) and (23) yield

$$\xi + \eta \equiv k, \xi \equiv \xi_1 + P, \eta \equiv \eta_1 + P, \zeta \equiv \zeta_1 + T, \phi \equiv \phi_1 + rT + Q \pmod{2}.$$

1) Take $k=1$. Then $w \equiv 1 \pmod{2}$.

2) Take $k=2$. We may take T, P, Q modulo 2 such that $2/\xi, 2/\zeta$ or $2/\zeta$. Then $2/\eta$ and $w \equiv 2$ or $0 \pmod{4}$ according as $2/\xi$ or $2/\zeta$.

Theorem 51. Every null $F \equiv Ax^2 + By^2 + 2C'z^2 + 8D'u^2, 2/ABC'D', A+B \equiv 2C' \pmod{8}$, satisfying (12) and (13) is universal.

Proof: Theorem 31 asserts the existence of solutions of $F=0$ such that $\pi = 2^K \pi', 2/\pi'$, contains no odd prime factors p if p^2 divides any one of A, B, C', D' . By Theorem 18, $w \equiv g \pmod{\pi'}$ is solvable for arbitrary g when π' is a product of distinct odd primes p, p^2 not dividing any one of A, B, C', D' . Let x_1, y_1, z_1, u_1 be a proper solution of $F=0$ such that $\pi = 2^K \pi', \pi'$ odd and containing no prime factors p if p^2 divides any one of A, B, C', D' . Then, according to Theorem 18, $w \equiv g \pmod{\pi'}$ is solvable for arbitrary g . It remains to determine T, P, Q, k modulo 2, 4, 8 such that $w \equiv g \pmod{2\pi k}$ is solvable for arbitrary g .

I. $K=1$. We may write $x_1 = 2x', y_1 = 2y', z_1 = 4z', \pi = 2\pi', 2/x'y'\pi'$. Then $2/\sigma LMR, 4/\tau$. (15) and (23) yield $\xi + \eta \equiv k, \xi \equiv \xi_1 + P, \eta \equiv \eta_1 + P, \zeta \equiv \zeta_1 + nT + Q, \phi \equiv \phi_1 + rT + nQ \pmod{2}$. Take $k=2k', 2/k'$. We may take T, P, Q modulo 2 such that $2/\eta\zeta$. Then $2/\xi$ and $w \equiv 4 \pmod{8}$. Therefore $X_1 = \frac{1}{4} X$,

$Y_1 = \frac{1}{4} Y$, $Z_1 = \frac{1}{4} Z$, $U_1 = \frac{1}{4} U$ are solutions of $F=0$ such that $X_1 \equiv Y_1 \equiv 0$, $Z_1 \equiv 2 \pmod{4}$, $2 \nmid U_1$, where X , Y , Z , U are defined by (24). It remains for us to consider

II. $K=2$. We may write $x_1 = 4x'$, $y_1 = 4y'$, $z_1 = 2z'$, $\pi = 4\pi'$, $2 \nmid z' \pi'$. Then $2 \nmid \tau Nn$, $R \equiv 2 \pmod{4}$. (15) and (23) yield

$$x'\xi + y'\eta + \zeta \equiv k, \xi \equiv \xi_1 + \lambda T + \mu P, \eta \equiv \eta_1 + mT + lP, \zeta \equiv \zeta_1 + \sigma T, \\ \phi \equiv \phi_1 + r\sigma T + Q \pmod{2}.$$

1) $2 \nmid x'y'$ implies $2 \nmid \sigma LM$.

a) Take $k=1$.

1) We may take T , P modulo 4 such that $2 \nmid \xi$, $\eta \equiv 2$ or $0 \pmod{4}$. Then $2 \nmid \zeta$ and $w \equiv A+4$ or $A \pmod{8}$ according as $\eta \equiv 2$ or $0 \pmod{4}$.

2) We may take T , P modulo 4 such that $2 \nmid \xi$, $2 \nmid \eta$ and $\xi \equiv \eta$ or $\not\equiv \eta \pmod{4}$. Then $2 \nmid \zeta$ and $w \equiv 2C'$ or $2C'+4 \pmod{8}$ according as $\xi \equiv \eta$ or $\not\equiv \eta \pmod{4}$.

3) We may take T , P modulo 2 such that $2 \nmid \xi \eta$. Then $2 \nmid \zeta$ and $w \equiv 4 \pmod{8}$.

b) Take $k=2$.

1) We may take T , P , Q modulo 8 such that $\xi \equiv \pm 1$ or $\pm 3 \pmod{8}$, $\eta \equiv 2$ or $0 \pmod{4}$, $2 \nmid \phi$. Then $2 \nmid \zeta$. $w \equiv A+4B+2C'+8$ or $A+4B+2C' \pmod{16}$ when $\eta \equiv 2 \pmod{4}$ according as $\xi \equiv \pm 1$ or $\pm 3 \pmod{8}$. $w \equiv A+2C'+8$ or $A+2C' \pmod{16}$ when $4 \nmid \eta$ according as $\xi \equiv \pm 1$ or $\pm 3 \pmod{8}$.

2) We may take T , P , Q modulo 4 such that $\xi \equiv \eta \equiv 2$ or $0 \pmod{4}$, $2 \nmid \phi$. Then $4 \nmid \zeta$ and $w \equiv 0$ or $8 \pmod{16}$ according as $\xi \equiv \eta \equiv 2$ or $0 \pmod{4}$.

2) $2/x', 2/y'$ implies $2/\sigma LX, 2/M$.

a) Take $k=1$.

1) We may take T, P modulo 4 such that $2/\xi$, $\eta \equiv 2$ or $0 \pmod{4}$. Then $2/\zeta$ and $w \equiv A+4$ or $A \pmod{8}$ according as $\eta \equiv 2$ or $0 \pmod{4}$.

2) We may take T, P modulo 4 such that $2/\xi$, $2/\eta$ and $\xi \equiv \eta$ or $\not\equiv \eta \pmod{4}$. Then $2/\zeta$ and $w \equiv 2C'$ or $2C'+4 \pmod{8}$ according as $\xi \equiv \eta$ or $\not\equiv \eta \pmod{4}$.

b) Take $k=2$.

1) We may take T, P, Q modulo 8 such that $\xi \equiv \pm 1$ or $\pm 3 \pmod{8}$, $\eta \equiv 2$ or $0 \pmod{4}$, $2/\phi$. Then $2/\zeta$. $w \equiv A+4B+2C'+8$ or $A+4B+2C' \pmod{16}$ when $\eta \equiv 2 \pmod{4}$ according as $\xi \equiv \pm 1$ or $\pm 3 \pmod{8}$. $w \equiv A+2C'+8$ or $A+2C' \pmod{16}$ when $4/\eta$ according as $\xi \equiv \pm 1$ or $\pm 3 \pmod{8}$.

2) We may take T, P, Q modulo 4 such that $\xi \equiv \eta \equiv 2$ or $\xi \equiv 2, \eta \equiv 0 \pmod{4}$, $2/\phi$. Then $\zeta \equiv 2 \pmod{4}$ and $w \equiv 8$ or $4A \pmod{16}$ according as $\xi \equiv \eta \equiv 2$ or $\xi \equiv 2, \eta \equiv 0 \pmod{4}$.

c) Take $k=4$.

1) We may take T, P, Q modulo 8 such that $\xi \equiv 2 \pmod{4}$, $\eta \equiv 4$ or $0 \pmod{8}$, $2/\phi$. Then $\zeta \equiv 2 \pmod{4}$ and $w \equiv 4A+8C'+16$ or $4A+8C' \pmod{32}$ according as $\eta \equiv 4$ or $0 \pmod{8}$.

2) We may take T, P, Q modulo 8 such that $8/\xi$, $\eta \equiv 4$ or $0 \pmod{8}$, $2/\phi$. Then $4/\zeta$ and $w \equiv 16$ or $0 \pmod{32}$ according as $\eta \equiv 4$ or $0 \pmod{8}$.

3) $2/x', 2/y'$ imply $2/\sigma$. We may assume $x'y' \neq 0$.

a) Take $k=1$. Then $2 \nmid \xi$.

1) We may take T, P modulo 4 such that $2 \nmid \xi$, $\eta \equiv 2$ or $0 \pmod{4}$ since $2 \nmid \tau$ and $L\tau + M\eta = 1$. Then $w \equiv A + 2C'$ or $A + 2C' \pmod{8}$ according as $\eta \equiv 2$ or $0 \pmod{4}$.

2) We may take T, P modulo 4 such that $2/\xi$, $2/\eta$, $\xi \equiv \eta$ or $\not\equiv \eta \pmod{4}$. Then $w \equiv 2C'$ or $2C' + 4 \pmod{8}$ according as $\xi \equiv \eta$ or $\not\equiv \eta \pmod{4}$.

b) Take $k=2$. Then $2/\xi$.

1) We may take T, P, Q modulo 8 such that $\xi \equiv \pm 1$ or $\pm 3 \pmod{8}$, $\eta \equiv 2$ or $0 \pmod{4}$, $2 \nmid \phi$. Then

1.1) $x' \equiv 2 \pmod{4}$ implies $\xi \equiv 2 \pmod{4}$. $w \equiv A + 4B$ or $A + 4B + 8 \pmod{16}$ when $\eta \equiv 2 \pmod{4}$ according as $\xi \equiv \pm 1$ or $\pm 3 \pmod{8}$. $w \equiv A$ or $A + 8 \pmod{16}$ when $4/\eta$ according as $\xi \equiv \pm 1$ or $\pm 3 \pmod{8}$.

1.2) $4/x'$ implies $4/\xi$. $w \equiv A + 4B + 8$ or $A + 4B \pmod{16}$ when $\eta \equiv 2 \pmod{4}$ according as $\xi \equiv \pm 1$ or $\pm 3 \pmod{8}$. $w \equiv A + 8$ or $A \pmod{16}$ when $4/\eta$ according as $\xi \equiv \pm 1$ or $\pm 3 \pmod{8}$.

2) We may take T, P, Q modulo 4 such that $2/\xi$, $2/\eta$, $\xi \equiv \eta$ or $\not\equiv \eta \pmod{4}$, $2 \nmid \phi$. Then $4/\xi$. $w \equiv 8$ or $4A + 8 \pmod{16}$ according as $\xi \equiv \eta \equiv 0$ or $\xi \equiv 2, \eta \equiv 0 \pmod{4}$. $16/w$ when $\xi \equiv \eta \equiv 2 \pmod{4}$.

c) Take $k=4$.

1) We may take T, P, Q modulo 8 such that $\xi \equiv 2 \pmod{4}$, $8/\eta$, $2 \nmid \phi$. Then $\xi \equiv 2 \pmod{4}$ and $w \equiv 4A \pmod{32}$.

2) We may take T, P, Q modulo 8 such that

$\xi \equiv 2 \pmod{4}$, $\eta \equiv 4 \pmod{8}$, $2 \nmid \phi$. Then $\zeta \equiv 2 \pmod{4}$ and $w \equiv 4A + 16 \pmod{32}$.

Theorem 52. Every null $F \equiv Ax^2 + By^2 + 2C'z^2 + 2^e D'u^2$, $2 \nmid ABC'D'$, satisfying (9), (10), (11), (12), (13) is universal.

Proof: This follows as a consequence of Theorems 33, 43, 45, 46, 48, 50, 51.

6.3. JUST THREE OF A, B, C, D EVEN.

For the purposes of this section we may assume, without loss of generality, that $B = 2B'$, $C = 2^e C'$, $D = 2^f D'$, $2 \nmid AB'C'D'$, $e \geq 1$, $f \geq 1$. Then $F \equiv Ax^2 + 2B'y^2 + 2^e C'z^2 + 2^f D'u^2$. According to Theorem 4 there are no universal F having $e \geq 3$, $f \geq 3$. Therefore consider $e < 3$. Theorems 8 and 9 declare $f \leq 3$ and when $e = 1$, $f < 3$ if F is universal. Theorem 33 disposes of $e = f = 1$. The following cases remain to be studied:

$$F \equiv Ax^2 + 2B'y^2 + 2C'z^2 + 4D'u^2,$$

$$F \equiv Ax^2 + 2B'y^2 + 4C'z^2 + 4D'u^2,$$

$$F \equiv Ax^2 + 2B'y^2 + 4C'z^2 + 8D'u^2.$$

Theorem 53. Every null $F \equiv Ax^2 + 2B'y^2 + 2C'z^2 + 4D'u^2$, $2 \nmid AB'C'D'$, satisfying (12) and (13) is universal.

Proof: Theorem 31 asserts the existence of solutions of $F = 0$ such that π contains no odd prime factors p if p^2 divides any one of A , B' , C' , D' . Let x_1, y_1, z_1, u_1 be a proper solution of $F = 0$ such that $\pi = 2\pi'$, $2 \nmid \pi'$, contains no odd prime factors p if p^2 divides any one of A , B' , C' , D' . By Theorem 18, $w \equiv g \pmod{\pi'}$ is solvable for arbitrary g . It remains to determine T, P, Q, k modulo 2 or 4 such that $w \equiv g \pmod{2\pi'k}$ is solvable for arbitrary g . Let $x_1 = 2x'$.

I. $2 \nmid y, z$, implies $2 \nmid \sigma \tau MNn$, $2/R$. (15) and (23)
yield

$$x' \xi + \eta + \zeta \equiv k, \quad \xi \equiv \xi_1 + \lambda T + P, \quad \eta \equiv \eta_1 + mT + LP, \quad \zeta \equiv \zeta_1 + T, \\ \phi \equiv \phi_1 + rT + Q \pmod{2}.$$

1) $2 \nmid x'$ implies $2 \nmid L$.

a) Take $k=1$.

1) We may take T, P modulo 2 such that $2 \nmid \xi \eta$.
Then $2 \nmid \zeta$ and $w \equiv A \pmod{4}$.

2) We may take T, P modulo 2 such that $2/\xi$,
 $2/\eta$. Then $2 \nmid \zeta$ and $w \equiv 2 \pmod{4}$.

b) Take $k=2$.

1) We may take T, P, Q modulo 2 such that
 $2 \nmid \xi \eta$, $2 \nmid \phi$ or $2/\phi$. Then $2/\zeta$ and $w \equiv A + 2B' + 4$ or $A + 2B'$
 $\pmod{8}$ according as $2 \nmid \phi$ or $2/\phi$.

2) We may take T, P, Q modulo 4 such that
 $\xi \equiv 2 \pmod{4}$, $2/\eta$, $2 \nmid \phi$ or $2/\phi$. Then $2/\zeta$ and $w \equiv 0$ or 4
 $\pmod{8}$ according as $2 \nmid \phi$ or $2/\phi$.

2) $2/x'$ implies $2/L$, $2/m$.

a) Take $k=1$.

1) We may take T, P modulo 2 such that $2 \nmid \xi \eta$.
Then $2/\zeta$ and $w \equiv A + 2 \pmod{4}$.

2) We may take T, P modulo 2 such that $2/\xi$,
 $2/\eta$. Then $2 \nmid \zeta$ and $w \equiv \quad \pmod{4}$.

b) Take $k=2$.

1) We may take T, P, Q modulo 2 such that
 $2 \nmid \xi$, $2/\eta$, $2 \nmid \phi$ or $2/\phi$. Then $2/\zeta$ and $w \equiv A + 4$ or A accord-
ing as $2 \nmid \phi$ or $2/\phi$.

2) We may take T, P, Q modulo 4 such that $\xi \equiv 2 \pmod{4}$, $2/\eta$, $2/\phi$ or $2/\phi$. Then $2/\xi$ and $w \equiv 0$ or $4 \pmod{8}$ according as $2/\phi$ or $2/\phi$.

II. $2/y_1, 2/z_1$ imply $2/x_1 u, \sigma LRX, 2/M, \tau \equiv 2 \pmod{4}$.
(15) and (23) yield

$$\xi \equiv k, \eta \equiv \eta_1 + P, \zeta \equiv \zeta_1 + nT + Q, \phi \equiv \phi_1 + rT + nQ \pmod{2}.$$

1) Take $k=1$. Then $2/\xi$. We may take T, P, Q modulo 2 such that $2/\eta$, $2/\zeta$ or $2/\zeta$. Then $w \equiv A$ or $A+2 \pmod{4}$ according as $2/\zeta$ or $2/\zeta$.

2) Take $k=2$. Then $2/\xi$.

a) $y_1 \equiv 2 \pmod{4}$. Then $M \equiv 2 \pmod{4}$. We may take T, P, Q modulo 4 such that $4/\xi$, $2/\eta$, $2/\zeta$. Then $2/\phi$ and $w \equiv 2B' \pmod{8}$.

b) $4/y_1, 4/z_1$ imply $4/M, 2/N, 2/r$. We may take T, P, Q modulo 4 such that $\xi \equiv 2 \pmod{4}$, $2/\eta$, $2/\zeta$. Then $2/\phi$ and $w \equiv 2B' + 4 \pmod{8}$.

c) We may take T, P, Q modulo 4 such that $4/\xi$, $2/\eta$, $2/\zeta$. Then $2/\phi$ and $w \equiv 4 \pmod{8}$.

3) Take $k=4$. Then $2/\xi$.

a) $y_1 \equiv 2 \pmod{4}$. We may take T, P, Q modulo 4 such that $\xi \equiv 2$, $\zeta \equiv 0$ or $2 \pmod{4}$, $2/\eta$. Then $2/\phi$ and $w \equiv 4A + 2B'$ or $4A + 2B' + 8 \pmod{16}$ according as $\zeta \equiv 0$ or $2 \pmod{4}$.

b) $4/y_1, 4/z_1$. We may take T, P, Q modulo 4 such that $4/\xi$, $\zeta \equiv 2$ or $0 \pmod{4}$, $2/\eta$. Then $2/\phi$ and $w \equiv 2B' + 8$ or $2B' \pmod{16}$ according as $\zeta \equiv 2$ or $0 \pmod{4}$.

c) We may take T, P, Q modulo 4 such that $\xi \equiv 2$,

$\eta \equiv \zeta \equiv 0 \pmod{4}$. Then $2 \nmid \phi$ and $w \equiv 4A + 4D' \pmod{16}$.

d) We may take T, P, Q modulo 4 such that

$\xi \equiv \eta \equiv 2, \zeta \equiv 0 \pmod{4}$. Then $2 \nmid \phi$ and $w \equiv 4A + 4D' + 8 \pmod{16}$.

Theorem 54. Every null $F \equiv Ax^2 + 2B'y^2 + 4C'z^2 + 4D'u^2$, $2 \nmid AB'C'D'$, satisfying (12) and (13) is universal.

Proof: Theorem 31 asserts the existence of solutions of $F=0$ such that π contains no odd prime factors p if p^2 divides any one of A, B', C', D' . Let x_1, y_1, z_1, u_1 be a proper solution of $F=0$ such that $\pi = 2^K \pi'$, $2 \nmid \pi'$, contains no odd prime factors p if p^2 divides any one of A, B', C', D' . By Theorem 18, $w \equiv g \pmod{\pi'}$ is solvable for arbitrary g . It remains to determine T, P, Q, k modulo 2 or 4 such that $w \equiv g \pmod{2\pi k}$ is solvable for arbitrary g .

I. $K=2$. We may write $x_1 = 4x', y_1 = 2y', \pi = 4\pi'$, $2 \nmid z_1 u_1 \pi'$. We may assume $x_1, y_1 \neq 0$. Then $2 \nmid \tau NR$. (15) and (23) yield

$$\begin{aligned} x'\xi + y'\eta + \zeta + \phi &\equiv k, \quad \xi \equiv \xi_1 + lT + mP, \quad \eta \equiv \eta_1 + nT + lP, \\ \zeta &\equiv \zeta_1 + n\sigma T + Q, \quad \phi \equiv \phi_1 + r\sigma T + Q \pmod{2}. \end{aligned}$$

Take $k \equiv k', 2 \nmid k'$. We may take T, P, Q modulo 4 such that $\xi \equiv 2 \pmod{4}, 2/\eta, 2/\zeta$. Then $2 \nmid \phi$ and $8/w$. Therefore $X_1 = \frac{1}{8}X, Y_1 = \frac{1}{8}Y, Z_1 = \frac{1}{8}Z, U_1 = \frac{1}{8}U$ are solutions of $F=0$ such that $X_1 \equiv 2 \pmod{4}, 2/Y_1$ and just one of Z_1, U_1 odd, where X, Y, Z, U are defined by (24). It remains for us to consider

II. $K=1$. We may write $x_1 = 2x', y_1 = 2y', z_1 = 2z', \pi = 2\pi', 2 \nmid x'u_1 \pi'$. Then $2 \nmid \sigma LR/r, 2/M, 2/N, \tau \equiv 2 \pmod{4}$. (15) and (23) yield

$$\xi \equiv k, \quad \eta \equiv \eta_1 + P, \quad \zeta \equiv \zeta_1 + nT + Q, \quad \phi \equiv \phi_1 + T \pmod{2}.$$

1) Take $k=1$. Then $2 \nmid \xi$. We may take P modulo 2 such that $2/\eta$ or $2 \nmid \eta$. Then $w \equiv A$ or $A+2 \pmod{4}$ according as $2/\eta$ or $2 \nmid \eta$.

2) Take $k=2$. Then $2/\xi$.

a) $2/y'$ implies $4/M$. We may take T, P, Q modulo 4 such that $4/\xi$, $2 \nmid \eta$, $2 \nmid \zeta$ or $2/\zeta$. Then $2 \nmid \phi$ and $w \equiv 2B'$ or $2B'+4 \pmod{8}$ according as $2 \nmid \zeta$ or $2/\zeta$.

b) $2/y'$ implies $M \equiv 2 \pmod{4}$.

1) We may take T, P, Q modulo 4 such that $4/\xi$, $2 \nmid \eta$, $2/\zeta$. Then $2/\phi$ and $w \equiv 2B' \pmod{8}$.

2) We may take T, P, Q modulo 4 such that $\xi \equiv 2 \pmod{4}$, $2 \nmid \eta \zeta$. Then $2 \nmid \phi$ and $w \equiv 2B'+4 \pmod{8}$.

c) We may take T, P, Q modulo 4 such that $4/\xi$, $2/\eta$, $2/\zeta$ or $2 \nmid \zeta$. Then $2 \nmid \phi$ and $w \equiv 4$ or $0 \pmod{8}$ according as $2/\zeta$ or $2 \nmid \zeta$.

Theorem 55. Every null $F \equiv Ax^2 + 2B'y^2 + 4C'z^2 + 8D'u^2$, $2 \nmid AB'C'D'$, satisfying (12) and (13) is universal.

Proof: Theorem 31 asserts the existence of solutions of $F \equiv 0$ such that π contains no odd prime factors p if p^2 divides any one of A, B', C', D' . Let x_1, y_1, z_1, u_1 be a proper solution of $F \equiv 0$ such that $\pi = 2^K \pi'$, $2 \nmid \pi'$, contains no odd prime factors p if p^2 divides any one of A, B', C', D' . By Theorem 18, $w \equiv g \pmod{\pi'}$ is solvable for arbitrary g . It remains to determine T, P, Q, k modulo 2, 4, 8 such that $w \equiv g \pmod{2\pi k}$ is solvable.

I. $K=2$. We may write $x_1 = 4x'$, $y_1 = 2y'$, $z_1 = 2z'$, $2 \nmid y'u_1$. Then $\pi = 4\pi'$, $2 \nmid \pi'$, and $2 \nmid \sigma MR$, $\tau \equiv 2 \pmod{4}$. (15)

and (23) yield

$$\begin{aligned}x'\xi + \eta &\equiv k, \quad \xi \equiv \xi_i + P, \quad \eta \equiv \eta_i + LP, \quad \zeta \equiv \zeta_i + nT + Q, \\ \phi &\equiv \phi_i + rT + NQ \pmod{2}.\end{aligned}$$

1) $2/x'$ implies $2/L$, $2/\phi$. Take $k=2k'$, $2/k'$. We may take T, P, Q modulo 4 such that $\xi \equiv 2 \pmod{4}$, $\eta \equiv 2$ or $0 \pmod{4}$ according as $2/z'$ or $2/z'$, $2/\zeta$. Then $2/\phi$ and $8/w$. Therefore $X_i = \frac{1}{2^6} X$, $Y_i = \frac{1}{2^6} Y$, $Z_i = \frac{1}{2^6} Z$, $U_i = \frac{1}{2^6} U$ are solutions of $F=0$ such that $2/U_i$, $X_i \equiv 4 \pmod{8}$ when $\theta=3$, or $X_i \equiv 2 \pmod{4}$ when $\theta=4$, where X, Y, Z, U are defined by (24) and $\theta=3$ or 4 according as $w \equiv 8$ or $0 \pmod{16}$.

2) $2/x'$ implies $2/L$.

a) Take $k=1$.

1) We may take T, P, Q modulo 2 such that $2/\xi$, $2/\zeta$ or $2/\zeta$. Then $2/\eta$ and $w \equiv A$ or $A+4 \pmod{8}$ according as $2/\zeta$ or $2/\zeta$.

2) We may take T, P, Q modulo 4 such that $4/\xi$, $2/\zeta$ or $2/\zeta$. Then $2/\eta$ and $w \equiv 2B'$ or $2B'+4 \pmod{8}$ according as $2/\zeta$ or $2/\zeta$.

b) Take $k=2$.

1) We may take T, P, Q modulo 8 such that $\xi \equiv \pm 1$ or $\pm 3 \pmod{8}$, $2/\zeta$ or $2/\zeta$, $2/\phi$. Then $2/\eta$. $w \equiv A+2B'+4C'+8$ or $A+2B'+4C' \pmod{16}$ when $2/\zeta$ according as $\xi \equiv \pm 1$ or $\pm 3 \pmod{8}$. $w \equiv A+2B'+8$ or $A+2B' \pmod{16}$ when $2/\zeta$ according as $\xi \equiv \pm 1$ or $\pm 3 \pmod{8}$.

2) We may take T, P, Q modulo 4 such that $\xi \equiv 2 \pmod{4}$, $2/\zeta$, $2/\phi$. Then $4/\eta$ and $w \equiv 4A \pmod{16}$.

3) We may take T, P, Q modulo 4 such that $4/\xi, 2/\zeta, 2/\phi$. Then $\eta \equiv 2 \pmod{4}$ and $w \equiv 8 \pmod{16}$.

c) Take $k=4$.

1) We may take T, P, Q modulo 4 such that $\xi \equiv 2 \pmod{4}, \zeta \equiv 0 \text{ or } 2 \pmod{4}, 2/\phi$. Then $\eta \equiv 2 \pmod{4}$ and $w \equiv 4A + 8B' \text{ or } 4A + 8B' + 16 \pmod{32}$ according as $\zeta \equiv 0 \text{ or } 2 \pmod{4}$.

2) We may take T, P, Q modulo 8 such that $\xi \equiv 4 \pmod{8}, 4/\zeta, 4/\phi$. Then $4/\eta$ and $w \equiv 16 \pmod{32}$.

3) We may take T, P, Q modulo 8 such that $8/\xi, 4/\zeta, \phi \equiv 2 \pmod{4}$. Then $4/\eta$ and $32/w$.

II. $K=1$. We may write $x_i = 2x', y_i = 2y', \pi = 2\pi', 2/x'z, \pi'$. Then $2 \nmid \sigma \text{ LN} \nmid n, 2/M, 2/R, \tau \equiv 2 \pmod{4}$. (15) and (23) yield

$$\xi \equiv k, \eta \equiv \eta_1 + P, \zeta \equiv \zeta_1 + T, \phi \equiv \phi_1 + rT + Q \pmod{2}.$$

1) Take $k=1$. Then $2 \nmid \xi$. We may take P modulo 2 such that $2/\eta$ or $2 \nmid \eta$. Then $w \equiv A \text{ or } A+2 \pmod{4}$ according as $2/\eta$ or $2 \nmid \eta$.

2) Take $k=2$. Then $2/\xi$.

a) $2/y'$ implies $M \equiv 2 \pmod{4}$. We may take T, P modulo 4 such that $\xi \equiv 2 \pmod{4}, 2 \nmid \eta$. Then $2 \nmid \zeta$ and $w \equiv 2B' \pmod{8}$.

b) $2/y'$ implies $4/M$. We may take T, P modulo 4 such that $\xi \equiv 2 \pmod{4}, 2 \nmid \eta$. Then $2/\zeta$ and $w \equiv 2B' + 4 \pmod{8}$.

c) We may take T, P modulo 2 such that $\xi \equiv 2 \pmod{4}, 2/\eta$. Then $2/\zeta$ and $w \equiv 4 \pmod{8}$.

3) Take $k=4$. Then $2/\xi$.

a) $2 \nmid y'$. We may take T, P, Q modulo 4 such that $\xi \equiv 2 \pmod{4}$, $2 \nmid \eta$, $2/\phi$ or $2 \nmid \phi$. Then $2/\xi$ and $w \equiv 4A + 2B'$ or $4A + 2B' + 8 \pmod{16}$ according as $2/\phi$ or $2 \nmid \phi$.

b) $2/y'$. We may take T, P, Q modulo 4 such that $\xi \equiv 2 \pmod{4}$, $2 \nmid \eta$, $2/\phi$ or $2 \nmid \phi$. Then $2 \nmid \xi$ and $w \equiv 4A + 2B' + 4C'$ or $4A + 2B' + 4C' + 8 \pmod{16}$ according as $2/\phi$ or $2 \nmid \phi$.

c) $2 \nmid u$, implies $R \equiv 2 \pmod{4}$. We may take T, P, Q modulo 8 such that $8/\xi$, $4/\eta$, $2 \nmid \phi$ or $2/\phi$. Then $4/\xi$ and $w \equiv 8 \pmod{16}$ when $2 \nmid \phi$. $\xi \equiv 2 \pmod{4}$ and $16/w$ when $2/\phi$.

d) $2/u$, implies $4/R$. We may take T, P, Q modulo 8 such that $8/\xi$, $4/\eta$, $2 \nmid \phi$ or $2/\phi$. Then $\xi \equiv 2 \pmod{4}$ and $w \equiv 8$ or $0 \pmod{16}$ according as $2 \nmid \phi$ or $2/\phi$.

6.4. A SUPPLEMENTARY THEOREM.

In earlier sections we have assumed without proving

Theorem 56. If proper solutions of $F=0$ exist such that the corresponding π contain no odd prime factors p , p^2 a divisor of any one of A, B, C, D , and if proper solutions exist such that the corresponding $\pi = 2^K \pi'$, $2 \nmid \pi'$, then proper solutions exist such that the corresponding $\pi = 2^K \pi'$, $2 \nmid \pi'$, contain no odd prime factors p if p^2 divides any one of A, B, C, D .

Proof: Let x_1, y_1, z_1, u_1 be a proper solution of $F=0$ such that π contains no odd prime factors p if p^2 divides any one of A, B, C, D . If $\pi = 2^K \pi'$, $2 \nmid \pi'$, the theorem follows. Otherwise, from this solution on applying (24), we may construct proper solutions of $F=0$ such that $\pi = 2^K \pi'$, $2 \nmid \pi'$,

contains no odd prime factors p if p^2 divides any one of A , B , C , D . This is always possible since $2 \nmid p$.

7. CONCLUSION.

Now we are in a position to state our principal theorem.

Theorem 57. Every null F having properties (2)-(13) is universal.

Proof: The theorem follows as a consequence of Theorems 33, 39, 52, 53, 54, 55.

VITA

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